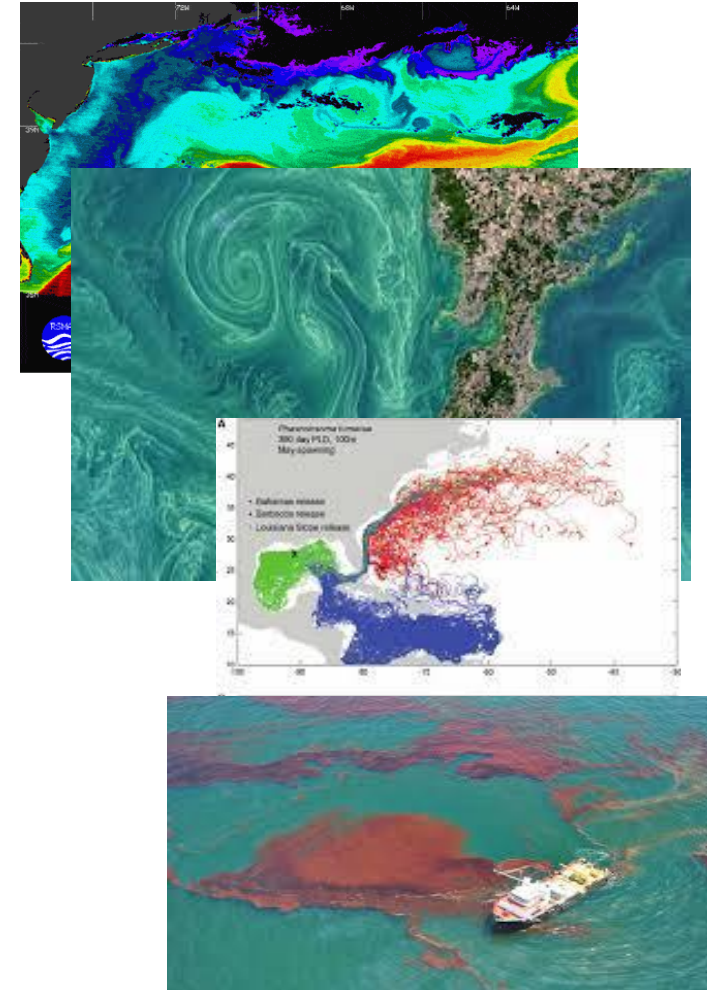


Lagrangian analysis tools for studying transport and mixing processes in ocean flows: fundamentals and applications

Ismael Hernández-Carrasco (IMEDEA, ICMAN, CSIC)

Transport and mixing processes: Why are they important?

- Central to study the **spatial and temporal structuring** of physical, chemical and biological variables.
- Control the **distribution of key climate tracers**: heat, temperature, salinity. Regulation of Earth's Climate.
- Are involved in biogeochemical cycles, nutrient distribution, biomass **heterogeneity**.
- Impact on the **marine ecosystems**, from microorganisms, larvae to top predators.
- Govern the **dispersion** (accumulation) processes of **pollutants**: debris, plastics, oil spills, etc.



Understanding how water moves between oceanic regions, and identifying the pathways and dynamical scales of this transport, is therefore of major theoretical and practical importance.

Lagrangian approach for studying transport and mixing

How to characterize these transport processes?

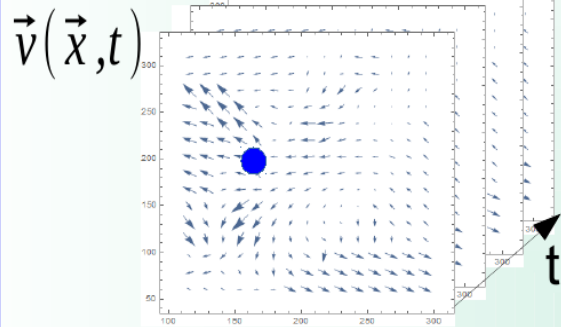
Equation for the evolution of a passive scalar field C

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = \nabla \cdot (D \nabla C)$$

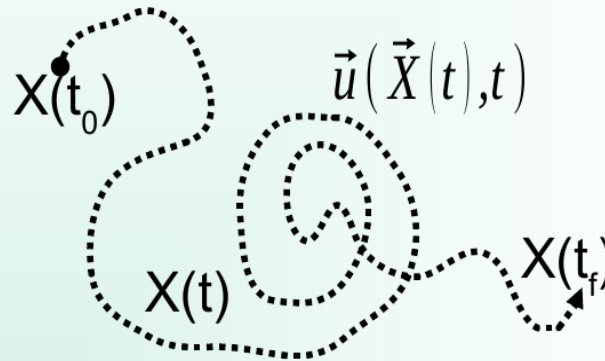
Lagrangian approach for studying transport and mixing

Both forms due to Euler (Lamb 1945 *Hydrodynamics*)

Eulerian: observe flow properties passing through a particular point in space
Velocity at all points of space occupied by the fluid, for all instants



Lagrangian: observe flow properties following fluid particles moving through the flow
Determine the history of any particle



How to characterize these transport processes?



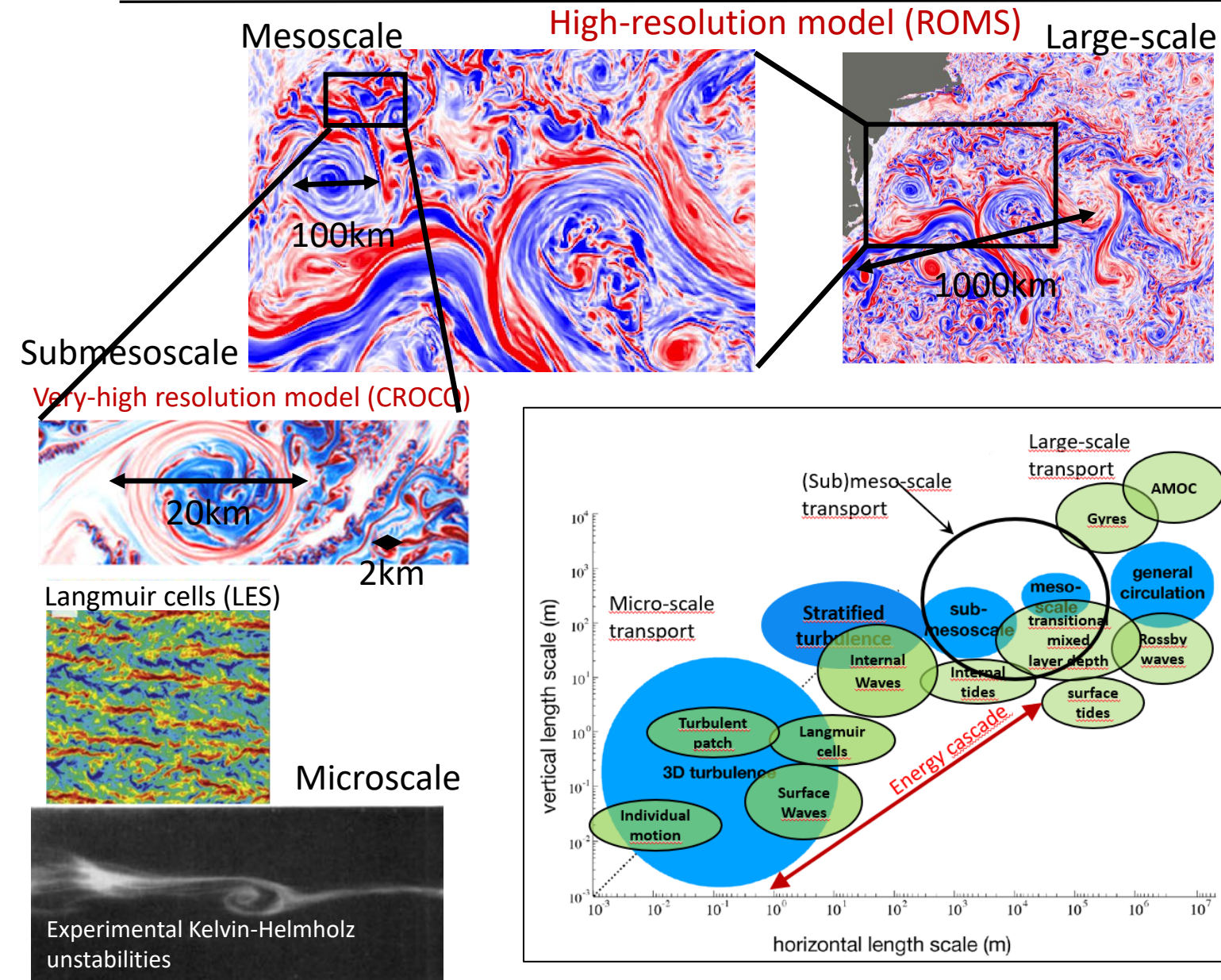
Lagrangian perspective

$$\vec{v}(\vec{x} = \vec{X}(\vec{X}_0, t), t) = \frac{d\vec{X}}{dt}(\vec{X}_0, t)$$

Lagrangian perspective

Natural framework to study the properties of transported particles, and therefore transport processes.

Multi-scale nonlinear dynamics



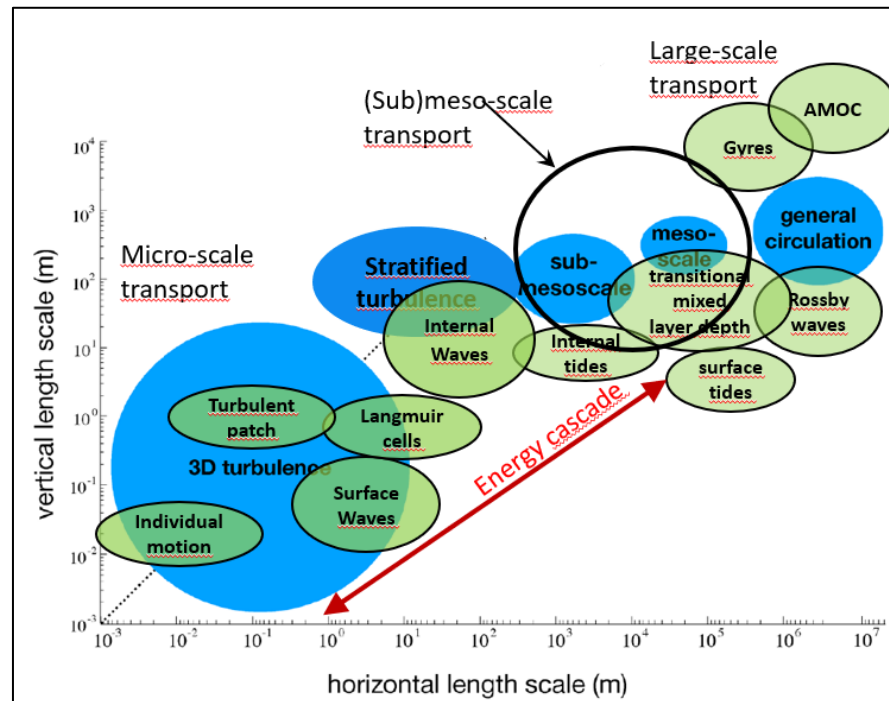
How to characterize these transport processes?



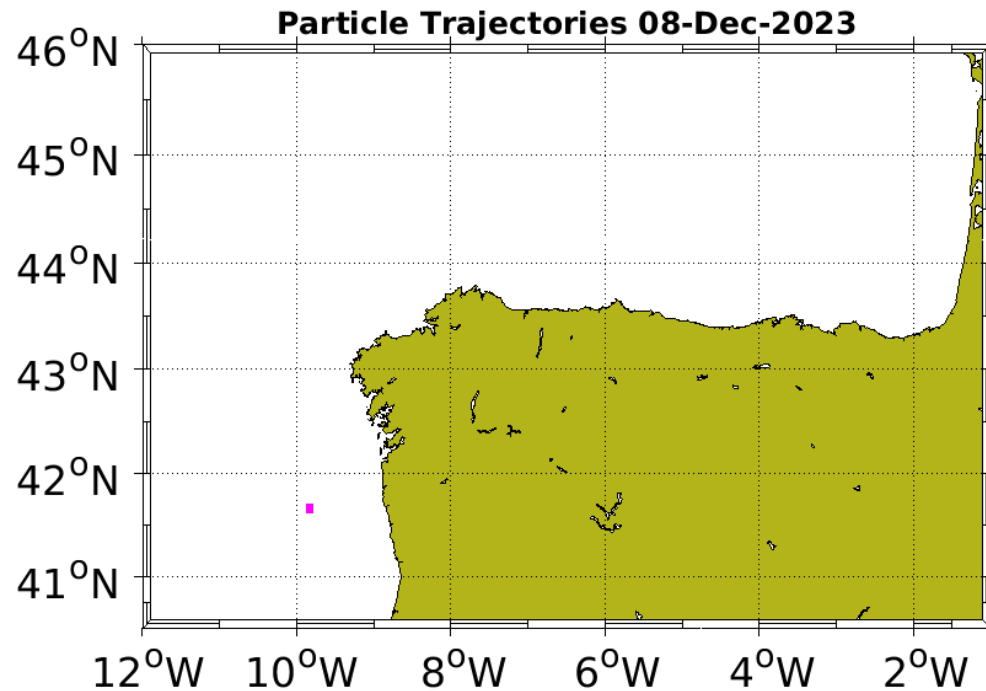
Lagrangian perspective



Nonlinear dynamics



Complex Lagrangian pathways: Lagrangian chaos



- The inherently non-linear nature of ocean flows leads to chaotic motion.
- Trajectories strongly dependent on initial conditions
- Large number of freedom degrees: 10^{24} in 3D flow (*Frisch 1995*).

How to characterize these transport processes?



Lagrangian perspective

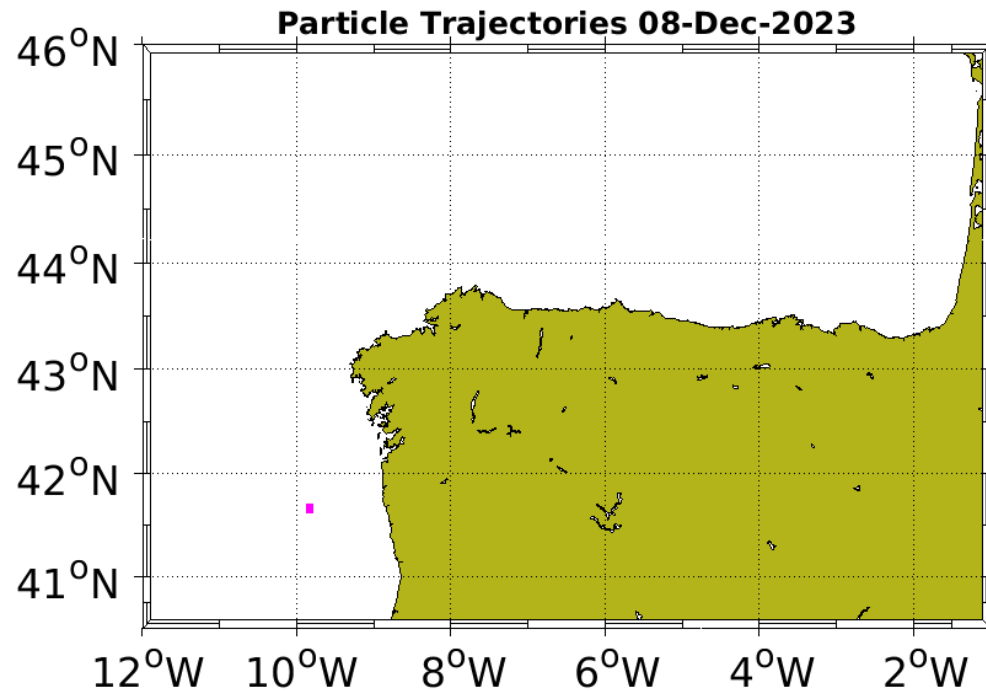


Nonlinear dynamics



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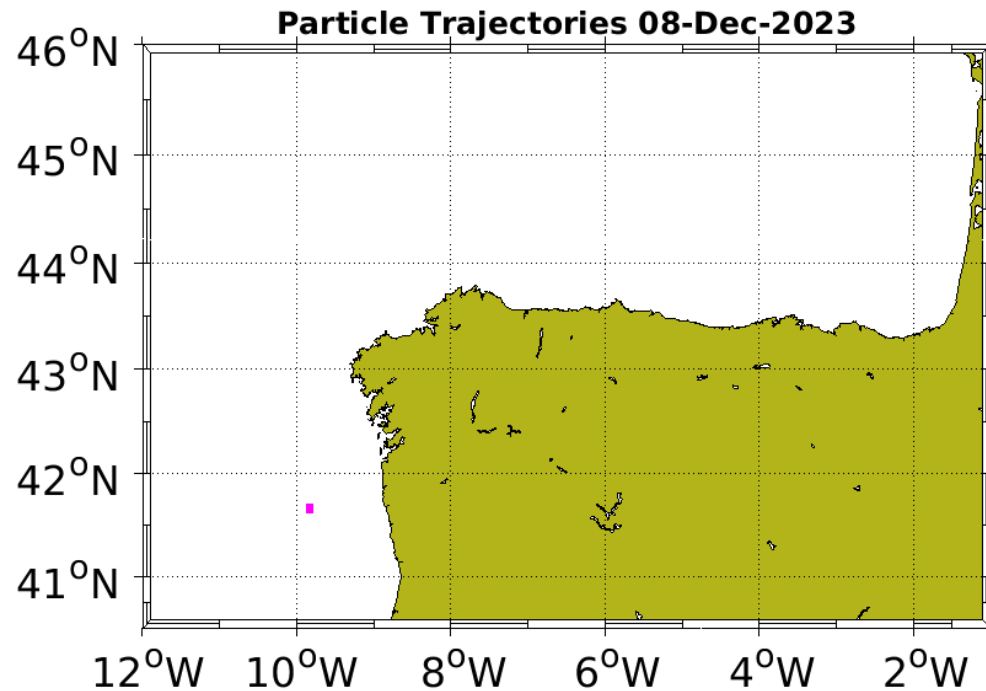


Lagrangian chaos

How can we reduce freedom degrees??

Which are the most relevant trajectories?

Dynamical system approach to Lagrangian transport



We address the problem of **Lagrangian chaos** (i.e. chaotic motion of fluid particles) using methods and tools borrowed from **the theory of dynamical systems**.

How to characterize these transport processes?



Lagrangian perspective



Nonlinear dynamics

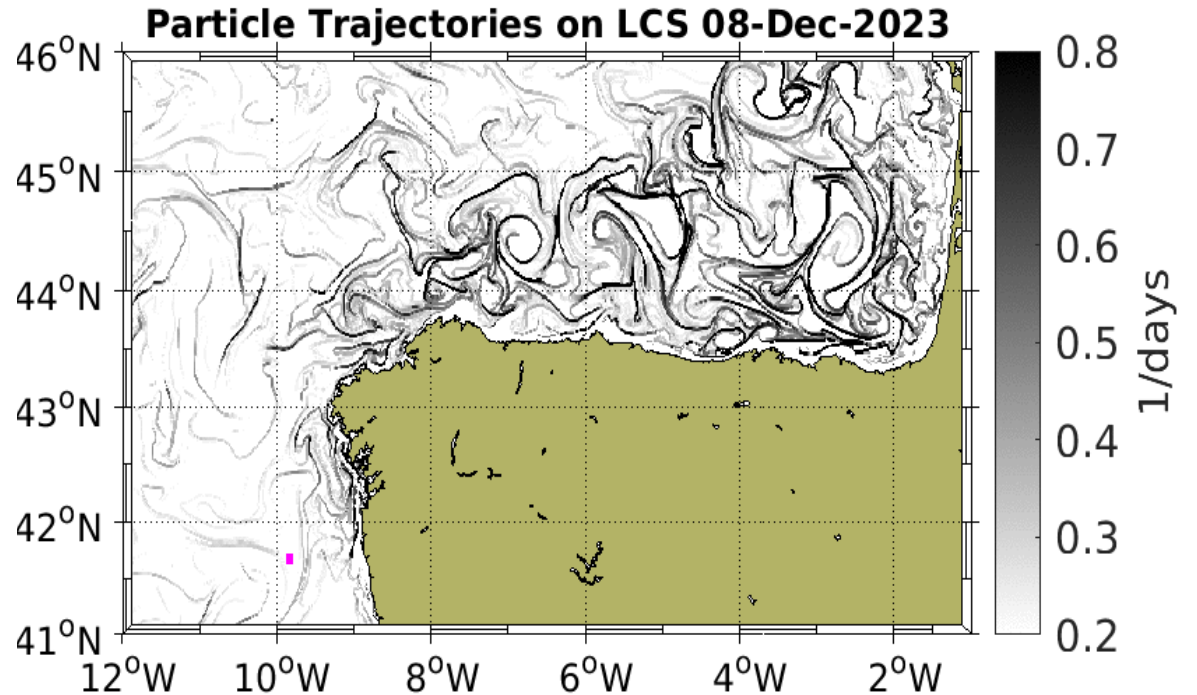
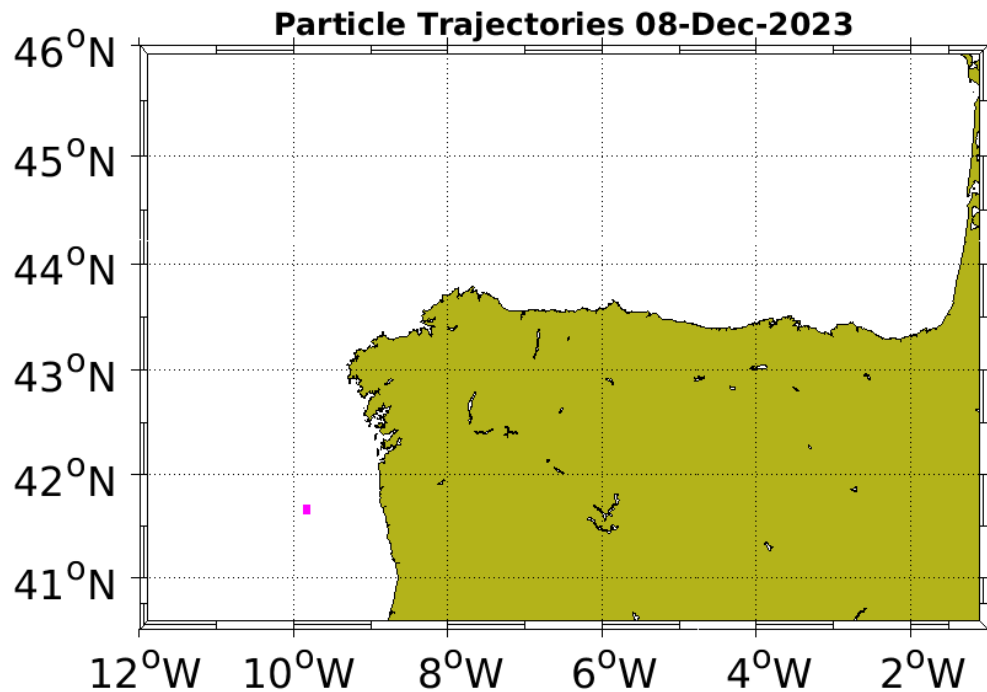


Lagrangian chaos



**Dynamical system
theory**

Lagrangian Coherent Structures



Among the most widely used Lagrangian approaches from dynamical systems theory concepts are the techniques well known as **Lagrangian Coherent Structures (LCS)**.

Outline

1. **Lagrangian coherent structures**

- Fundamentals
- Lagrangian analysis tools

2. **Stretching methods: Finite Size/Time Lyapunov Exponents**

- Definition and Properties
- Applications: characterization of ocean transport processes
- Applications: biophysical interactions

3. **Lagrangian averaged kinematic functions**

- Definition and Properties
- Applications: characterization of ocean transport processes.
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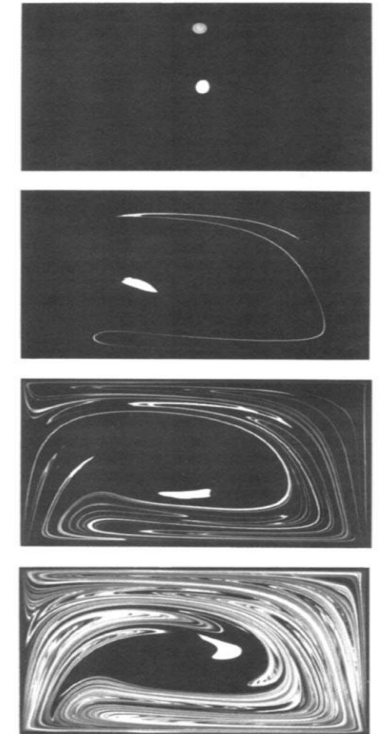
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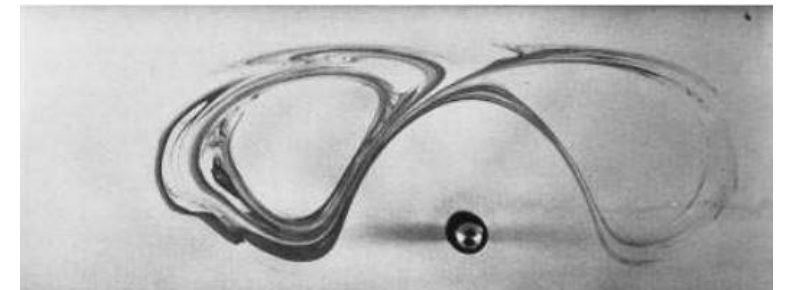
Experimental confirmation of chaotic motion

Ottino, 1990

Glycerine in Cavity flow



Rotor-oscillator cavity flow: manifolds

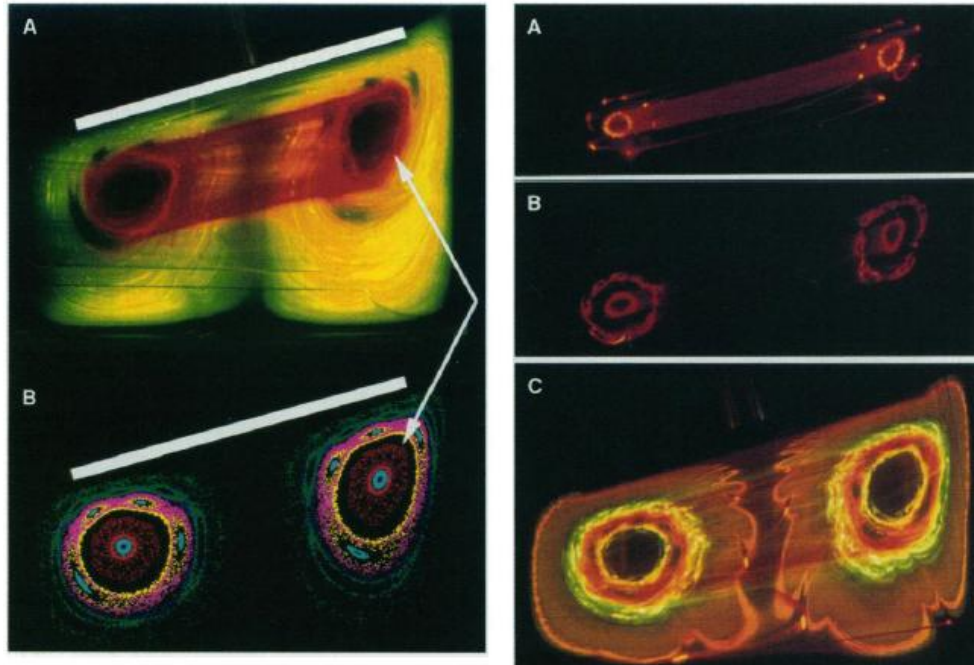


Hackborn, 1997

Batchelor, Ottino, Wiggins, Aref, Babiano, etc:

Mixing is intimately related to stretching and folding of material lines (or surfaces in three dimensions), and transport processes can be approached in terms of dynamical system description:

- Chaotic motion
- Manifolds
- KAM tori
- etc



(Fountain et al. 1998)

Chaotic picture captured in **3D experiments** where **fluorescent dye is advected in 3D flows in viscous fluid, cavity flows, etc.**

2D KAM tori (islands of regularity in **2D** cuts) are revealed clearly.

Dynamical system approach to Lagrangian transport

- **Lagrangian dynamical system:** Considering the set of particle trajectories advected in the velocity field as a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t))$$

- **2D Incompressibility:** symplectic geometry (Hamiltonian systems)

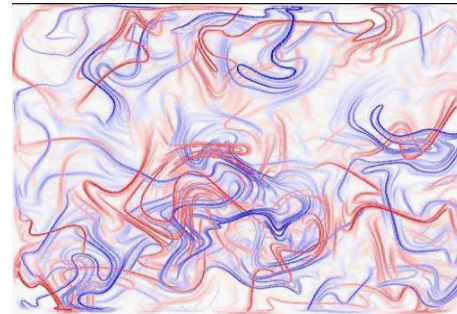
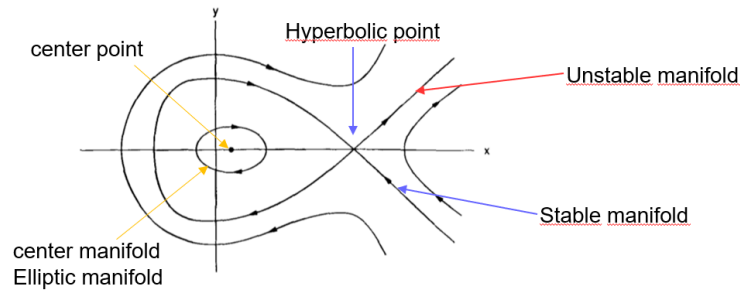
Streamfunction = Hamiltonian

$$\mathbf{V}_{\text{fluid}} \equiv (u(x, y, t), v(x, y, t)) = \left(\frac{\partial \psi}{\partial y}(x, y, t), -\frac{\partial \psi}{\partial x}(x, y, t) \right)$$

Hamilton's equation

$$\begin{aligned}\dot{x} &= \frac{\partial \psi}{\partial y}(x, y, t), \\ \dot{y} &= -\frac{\partial \psi}{\partial x}(x, y, t)\end{aligned}$$

- **Phase space = physical space** $(q, p) \longrightarrow (x, y)$



The goal of **global** theory of nonlinear dynamical systems is to study the **relationships** between “all” the trajectories of a dynamical system rather than seek ways of computing **individual trajectories**.

Global behavior in phase space is organized by relevant lines = Global flow geometry

WHICH ARE THESE RELEVANT LINES?

Ott, 1992
Aref, 1984
Ottino 1989
Babiano 1994
Wiggins 1992
Vulpiani 1990
Benzi 1990

Examples

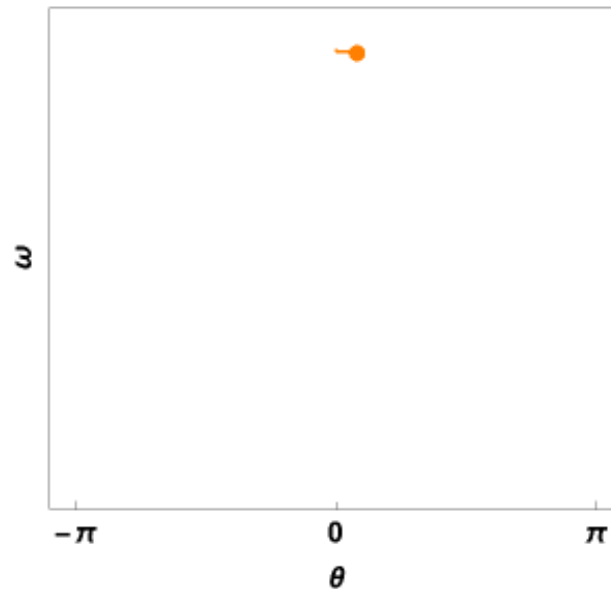
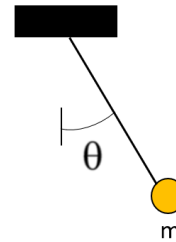
Time-independent Systems (autonomous). $\mathbf{v}$ is independent on time.

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t))$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

Example: Simple Pendulum

$$\frac{d^2\theta}{dt^2} = -mgl \sin \theta(t)$$



Manifolds: divide the flow into regions of different dynamics (**separatrices**)

Global flow geometry understood by studying **invariant manifolds** of the **fixed points**

- **Fixed points** $\mathbf{x}^c \longrightarrow \mathbf{v}(\mathbf{x}^c) = 0$

Oscillations: Periodic trajectories (**elliptic partition**)

Rotations: Hyperbolic trajectories (**hyperbolic partition**)

- **Manifolds (Separatrices)**

Stable manifold: trajectories asymptote $\mathbf{x}^c$ $t \rightarrow \infty$

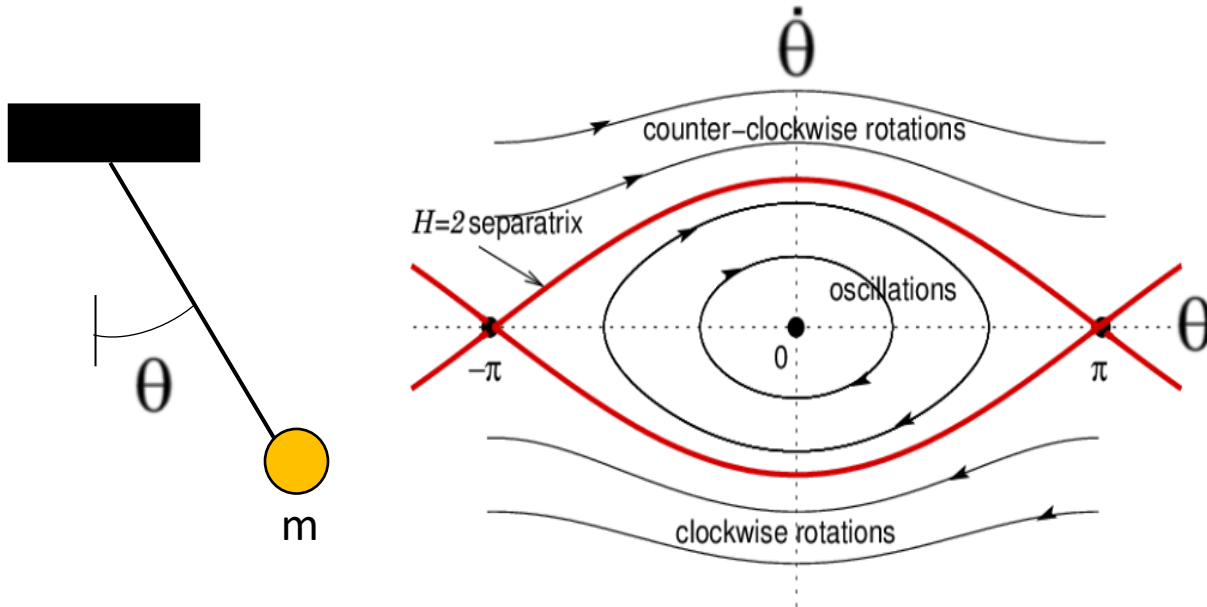
Unstable manifold: trajectories asymptote $\mathbf{x}^c$ $t \rightarrow -\infty$

Examples

Time-independent Systems (autonomous). $\mathbf{v}$ is independent on time.

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}(\mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}$$

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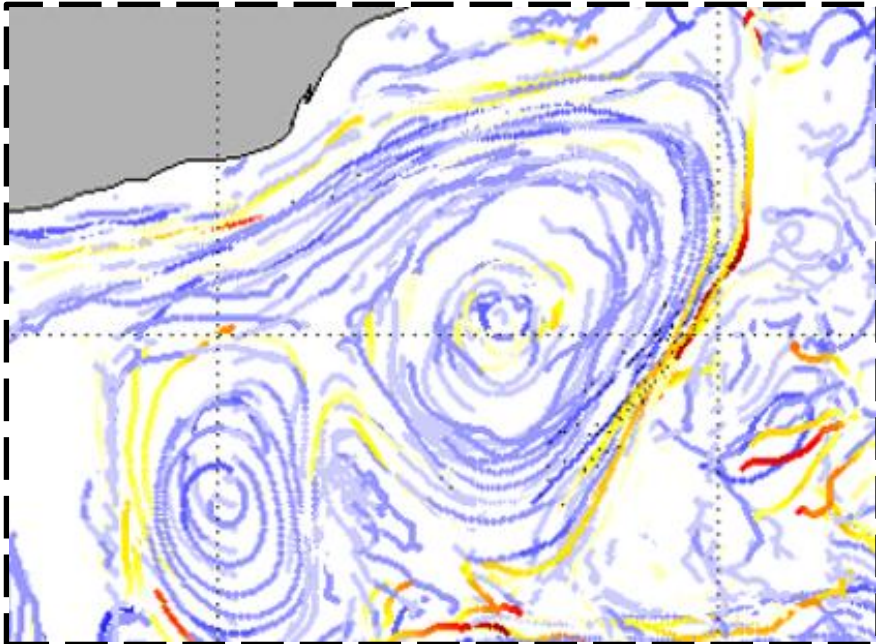
Dynamical system approach to ocean transport

But, ocean flows are unsteady, time-dependent systems ($\mathbf{v}$ depends on time)

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{v}(\mathbf{x}(t), t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

Phase space $\rightarrow$ Physical space

Fluid parcel trajectories from ROMS data



For time-dependent systems **manifolds** (separatrices) and **fixed points** change over time.

Hyperbolic point is generalized as **hyperbolic trajectory**.

Approach similar to hyperbolic points: instantaneous fixed points of autonomous systems.

Is there any particular subset of hyperbolic points and manifolds in oceanic flows organizing the dynamics (the equivalent to the fixed points in autonomous systems)?

How to select them among this mess of trajectories?

How to identify the separatrices?

Lagrangian analysis tools to identify Lagrangian Coherent Structures

Identifying the relevant trajectories and manifolds in time-aperiodic dynamical systems (ocean flows)

- **Distinguished hyperbolic trajectories** (Wiggins, Ide, Mancho, ...)
- **Stretching-field methods**: Finite-time Lyapunov exponents, Finite-size Lyapunov exponents, ...(Vulpiani, Boffeta, Legras, Haller, Lekien, Shaden, Hernandez-Carrasco, Beron-Vera...).
- **Lagrangian averaged functions**: Lagrangian descriptors, M Function, mesochronic (Mancho, Jiménez-Madrid, Mendoza, Mezic, Poje, Malhotra, Haller,...), Dilation rate, Finite-Time Lagrangian divergence, Finite-time Lagrangian vorticity (Haller, Huntley, Hernandez-Carrasco...,
- **Leaking methods**, escape, residence time methods (Tél, Schneider, ...)
- **Attracting or repelling material lines** (Haller, Poje, ...)
- **Coherent sets, Transfer operators** (Froyland et al., 2007, 2028, Denes et al.,2022)
- **Lagrangian flow networks** (Rossi et al., 2014, Ser-Giacomi, et al., 2015)
- Etc

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Outline

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- Dynamical system approach to Lagrangian transport
- Properties

2. **Stretching methods: Finite Size/Time Lyapunov Exponents**

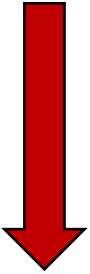
- Definition and Properties
- Applications: characterization of ocean transport processes
- Applications: biophysical interactions

3. Lagrangian averaged kinematic functions

- Definition and Properties
- Characterization of ocean transport processes.
- Applications: Biological impacts

Stretching methods: Lyapunov exponents

How to find separatrices?
We assume that separatrices divide regions of qualitatively different dynamics



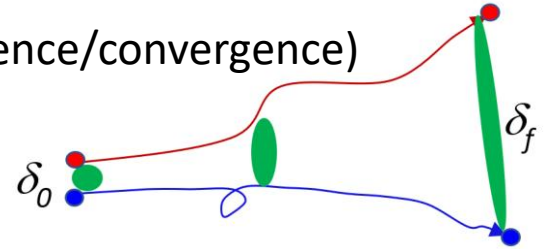
We quantify how dynamically different are two nearby trajectories, measuring divergence or stretching using **the concept of Lyapunov exponent**

Manifolds divide regions of qualitatively different dynamics

Separatrices: 

Exponential separation (divergence/convergence)

$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta \mathbf{x}(t_0) \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{x}(t)|}{|\delta \mathbf{x}(t_0)|}$$



Non-asymptotic Lyapunov Exponents

~~$$\lim_{t \rightarrow \infty} \lim_{\delta \mathbf{x}(t_0) \rightarrow 0}$$~~

Finite-time Lyapunov exponents

$$\lambda_{t_0}^{\tau}(\mathbf{x}_0) = \frac{1}{|\tau|} \ln \frac{|\delta \mathbf{x}(t_0 + \tau)|}{|\delta \mathbf{x}(t_0)|}$$

Haller 2001
Shadden et al, 2005

Finite-size Lyapunov exponents

$$\lambda_{\delta_0}^{\delta_f}(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \ln \frac{\delta_f}{\delta_0}$$

Aurell et al, 1996, 1997
Boffeta et al, 2001

Lagrangian Coherent Structures (LCS)

Stretching methods: Finite-Size Lyapunov Exponents

$$\lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \ln \frac{\delta_f}{\delta_0}$$

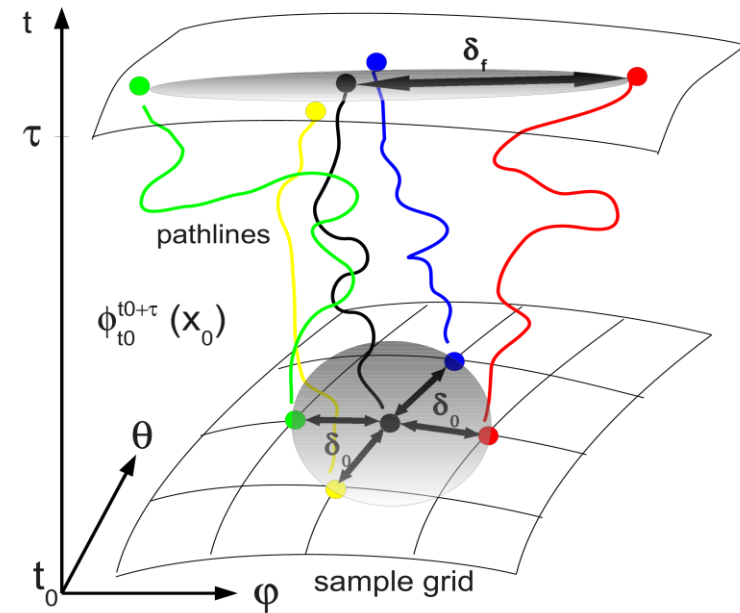
$\mathbf{x}$ are the initial coordinates of the particles

t_0 initial time for the integration

δ_0 is the initial separation.

δ_f is the final separation.

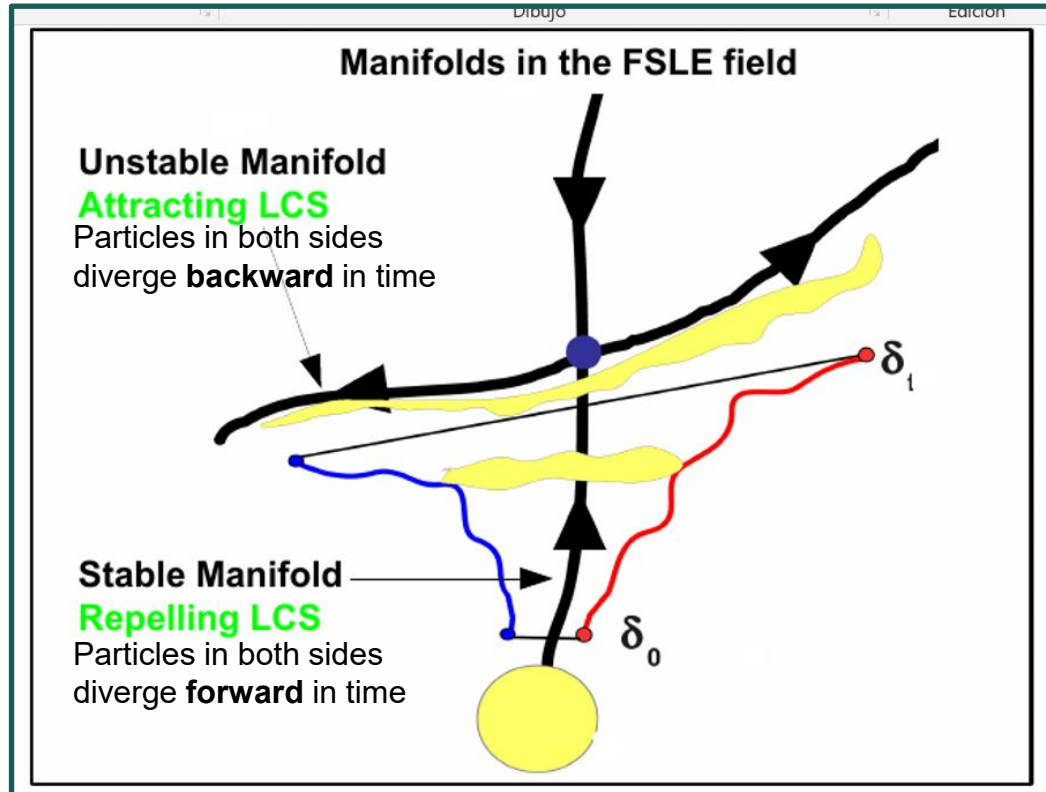
τ is the time needed for two particles initially separated δ_0 , to get separate δ_f



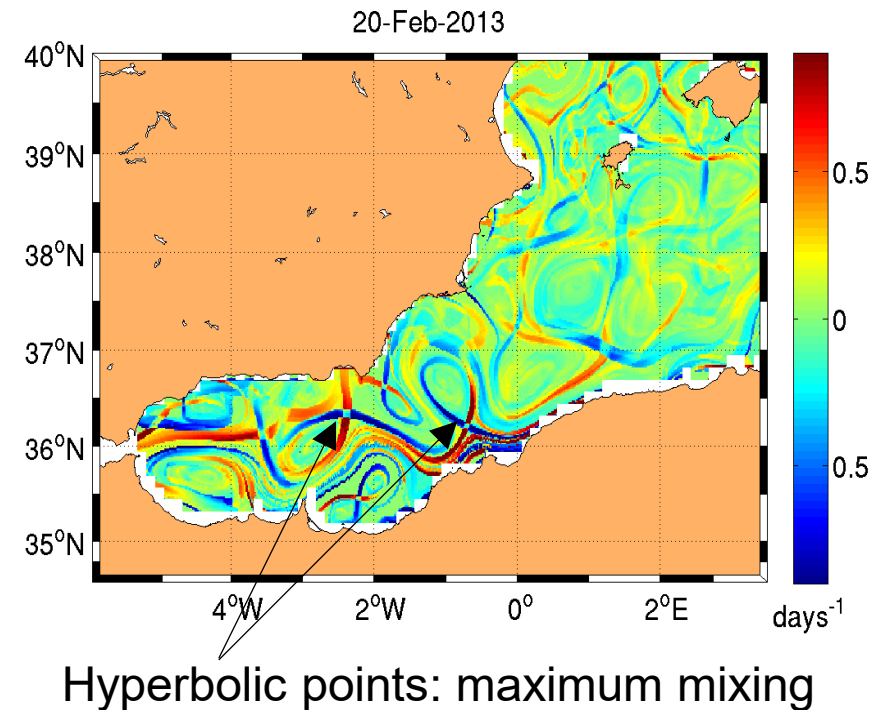
FSLE represent the inverse time for mixing up fluid parcels between scales δ_0 and δ_f

Stretching methods: Finite-Size Lyapunov Exponents

$$\lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \ln \frac{\delta_f}{\delta_0}$$



Forward – Backward FSLE

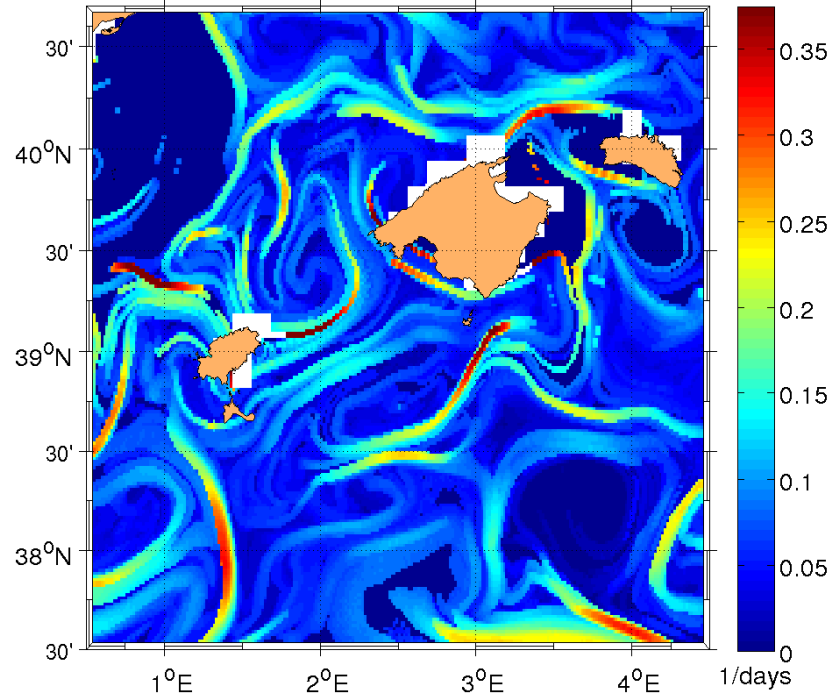


We compute the **Lagrangian Coherent Structures (LCS)** from ridges of the FSLE field (from high values of FSLE, which have a line-like shape)

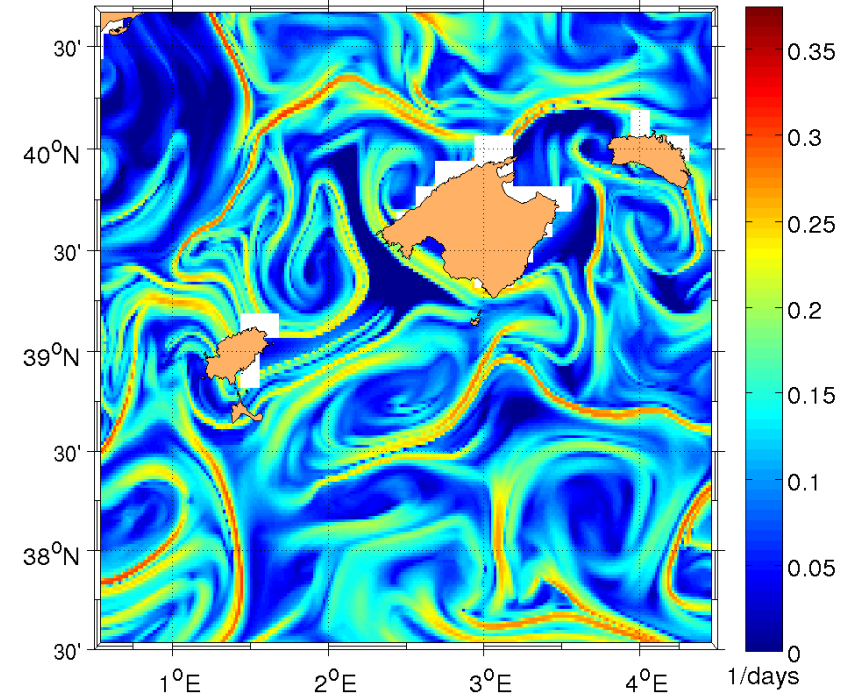
FTLE vs. FSLE

Altimetry – derived velocity fields

FSLE ($\Delta t_0 = 1/64\text{deg}$)
($\Delta t_f = 10 \cdot \Delta t_0$)

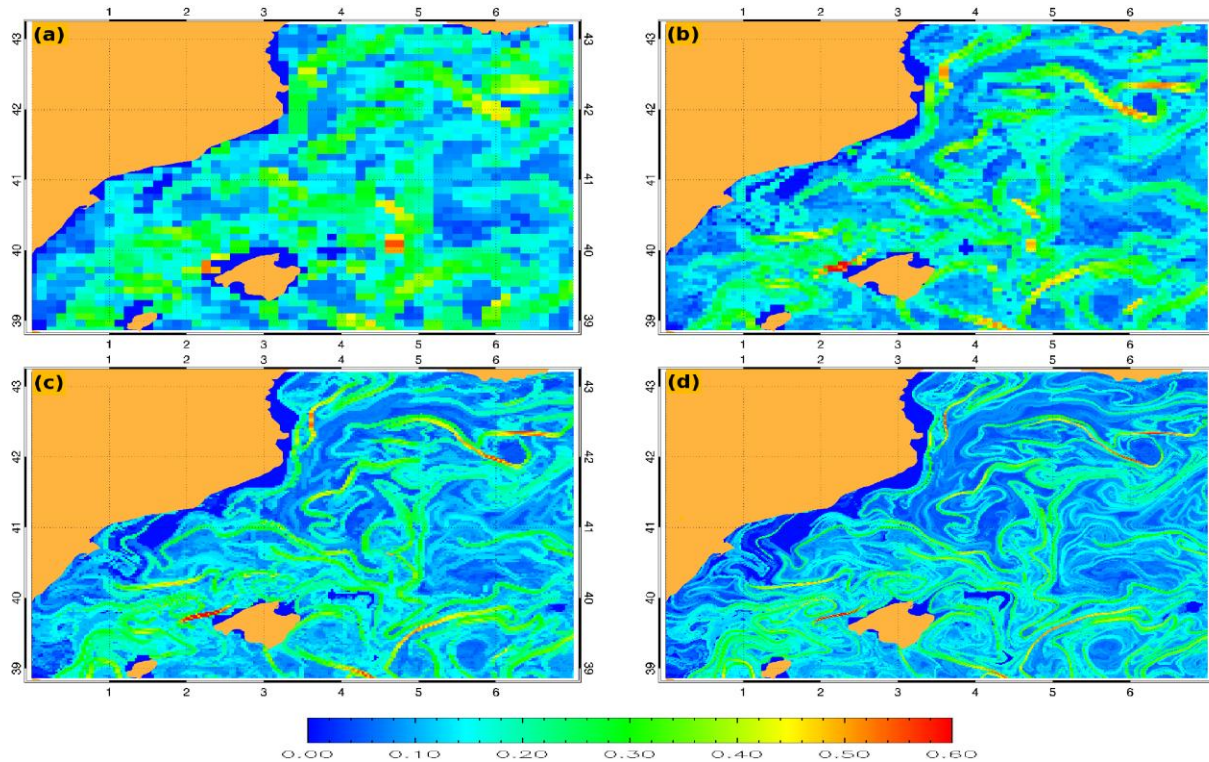


FTLE ($\Delta t_0 = 1/64\text{deg}$)
($T = 30\text{ days}$)



Properties of FSLE-LCS

Properties of LCS: Subgrid structures



a) $\delta_0 = 1/8^0$ $\Delta_0 = 1/8^0$

b) $\delta_0 = 1/16^0$ $\Delta_0 = 1/8^0$

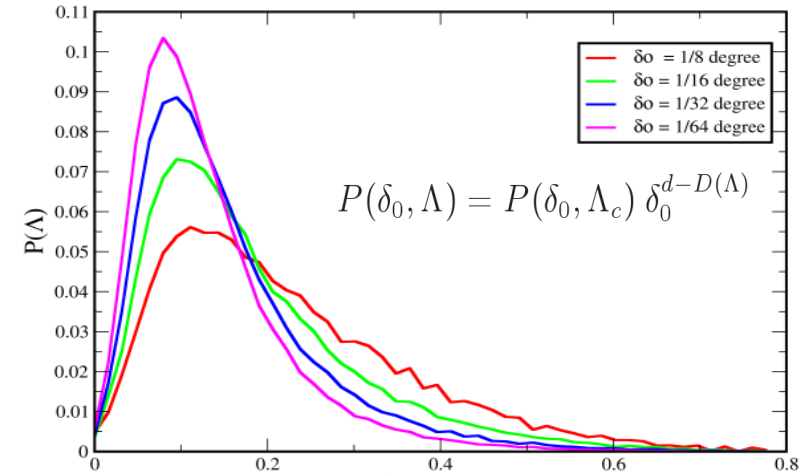
c) $\delta_0 = 1/32^0$ $\Delta_0 = 1/8^0$

d) $\delta_0 = 1/64^0$ $\Delta_0 = 1/8^0$

Note: δ_0 = resolution FSLE

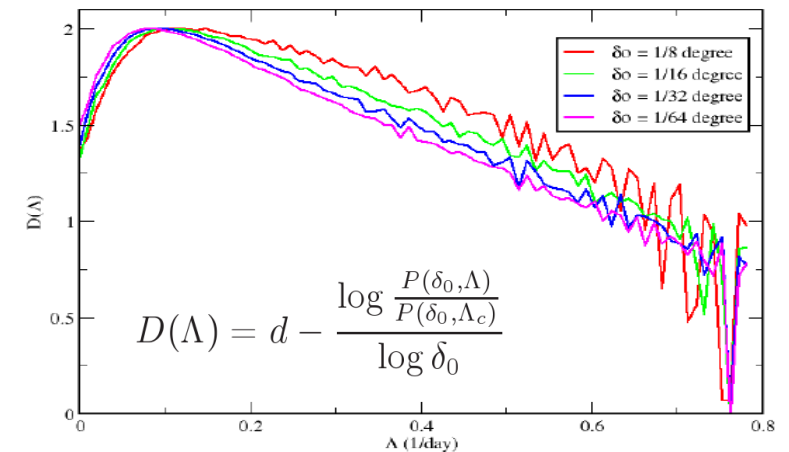
Increasing the spatial resolution of FSLE, the identification of the mesoscale and smaller (subgrid) structures is improved, while the large scale structures remain.

Histograms of FSLEs at different scales



$$P(\delta_0, \Lambda) = P(\delta_0, \Lambda_c) \delta_0^{d-D(\Lambda)}$$

Fractal dimension of FSLEs at different scales



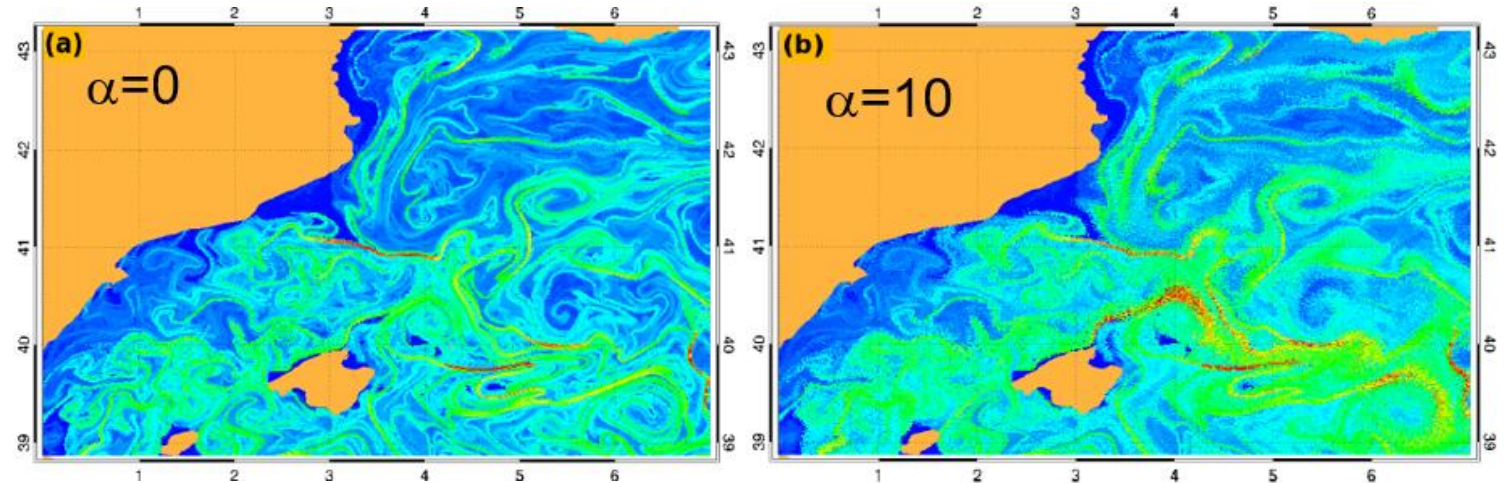
$$D(\Lambda) = d - \frac{\log \frac{P(\delta_0, \Lambda)}{P(\delta_0, \Lambda_c)}}{\log \delta_0}$$

Hernandez-Carrasco et al., 2011

Properties of LCS: Robustness of FSLE

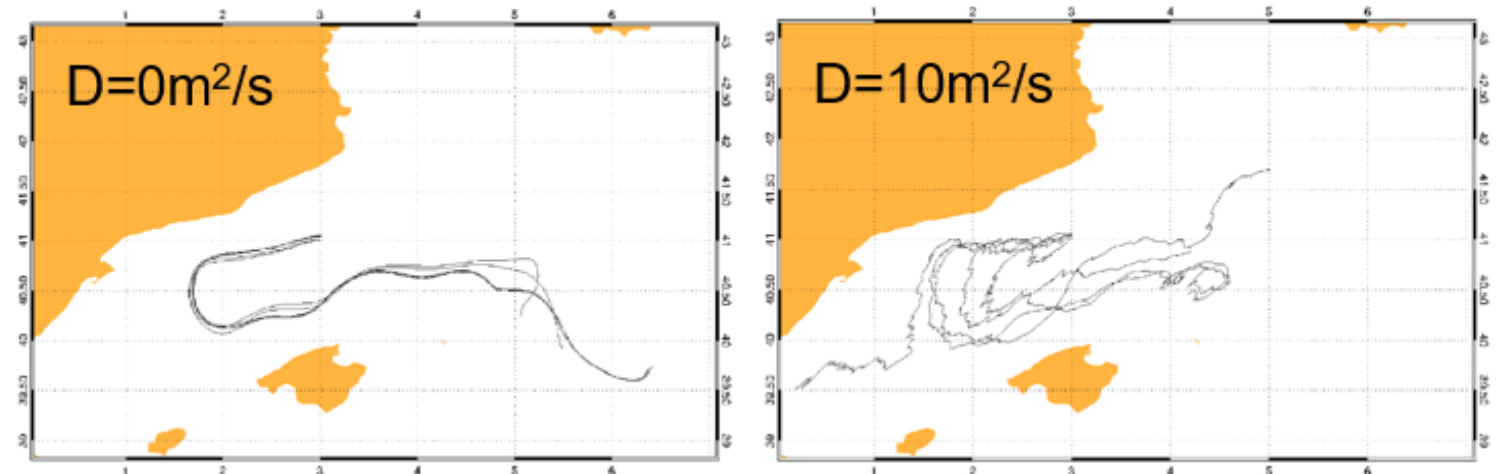
Effect of perturbed data

$$U'(\mathbf{x}; t) = u(\mathbf{x}; t)(1 + \alpha \varepsilon_x(\mathbf{x}; t))$$
$$V'(\mathbf{x}; t) = v(\mathbf{x}; t)(1 + \alpha \varepsilon_y(\mathbf{x}; t)).$$



Effect of dynamics of unsolved scales

$$\frac{d\phi}{dt} = \frac{u(\phi, \lambda, t)}{R \cos(\lambda)} + \frac{\sqrt{2D}}{R \cos(\lambda)} \xi_1(t)$$
$$\frac{d\lambda}{dt} = \frac{v(\phi, \lambda, t)}{R} + \frac{\sqrt{2D} \xi_2(t)}{R}$$



Hernandez-Carrasco et al., 2011

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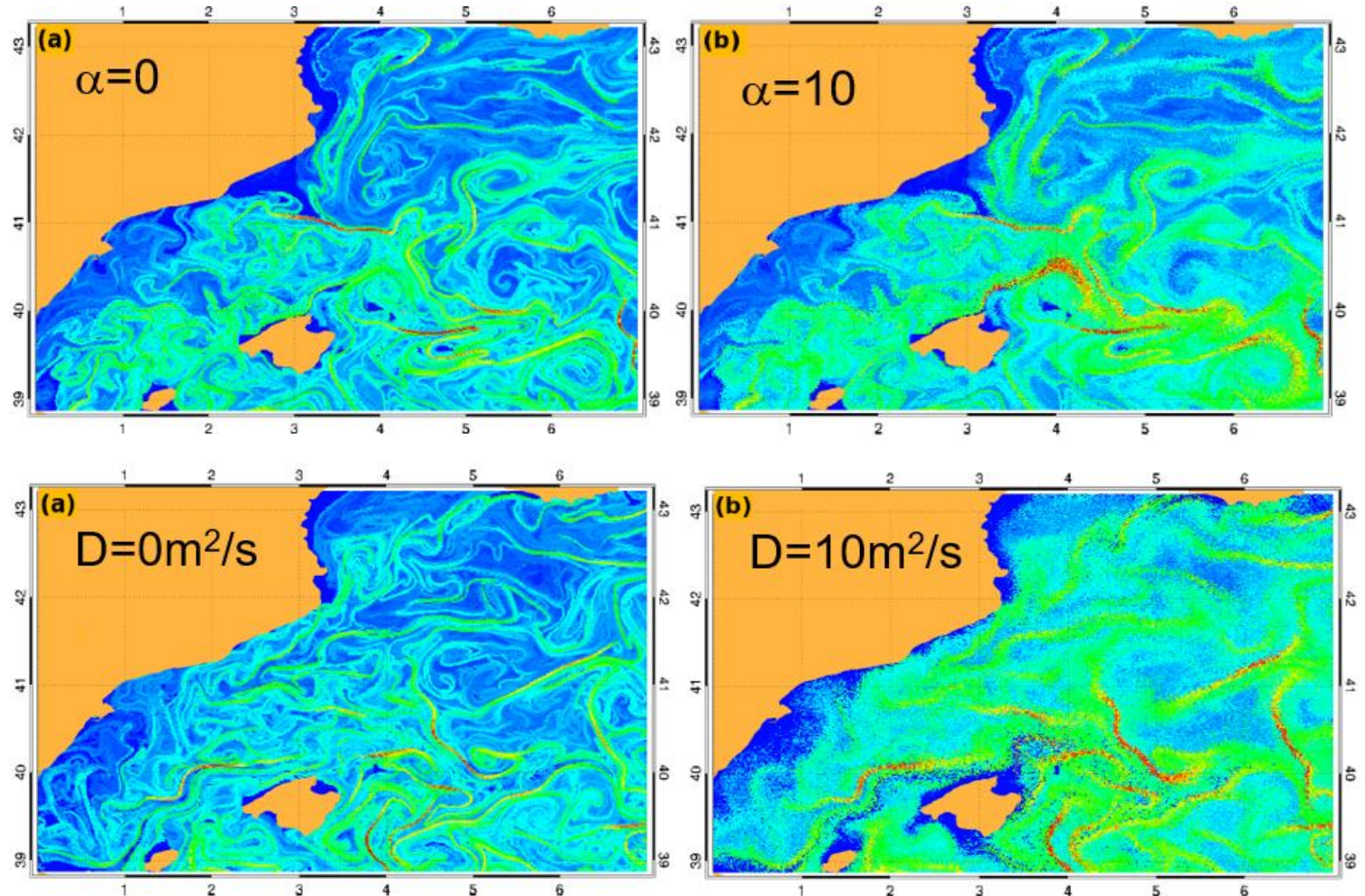
Gaussian random number

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$$\frac{d\phi}{dt} = \frac{u(\phi, \lambda, t)}{R \cos(\lambda)} + \frac{\sqrt{2D}}{R \cos(\lambda)} \xi_1(t)$$

$$\frac{d\lambda}{dt} = \frac{v(\phi, \lambda, t)}{R} + \frac{\sqrt{2D}\xi_2(t)}{R}$$

2D Gaussian white noise

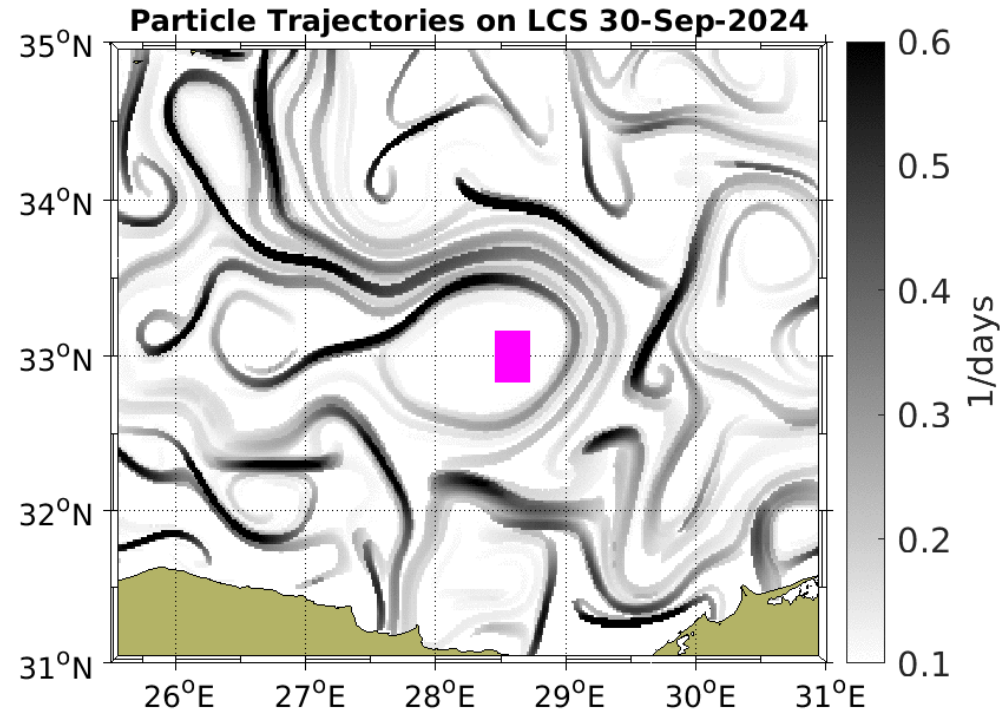


Hernandez-Carrasco et al., 2011

Even when some dynamics is missed the FSLEs results still give an accurate picture of the oceanic transport properties.

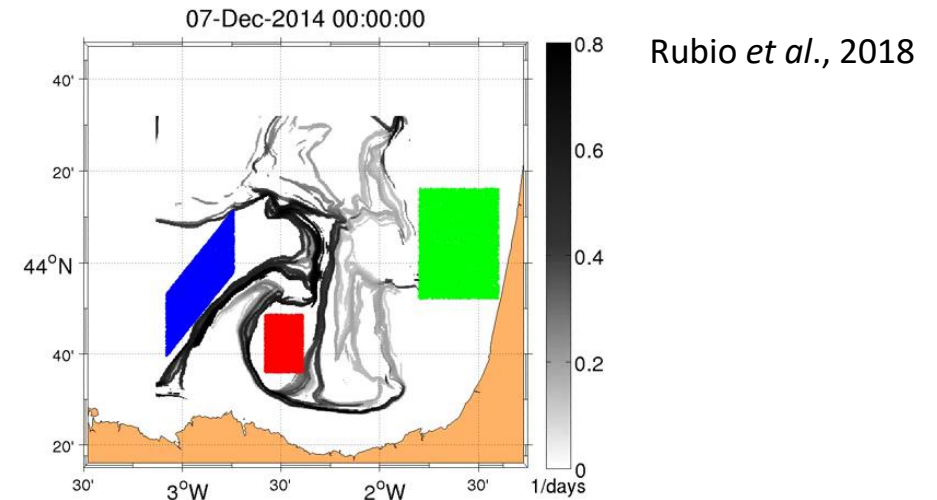
Properties of LCS: Transport barriers

Altimetry ($1/8^\circ$) (Mesoscale)



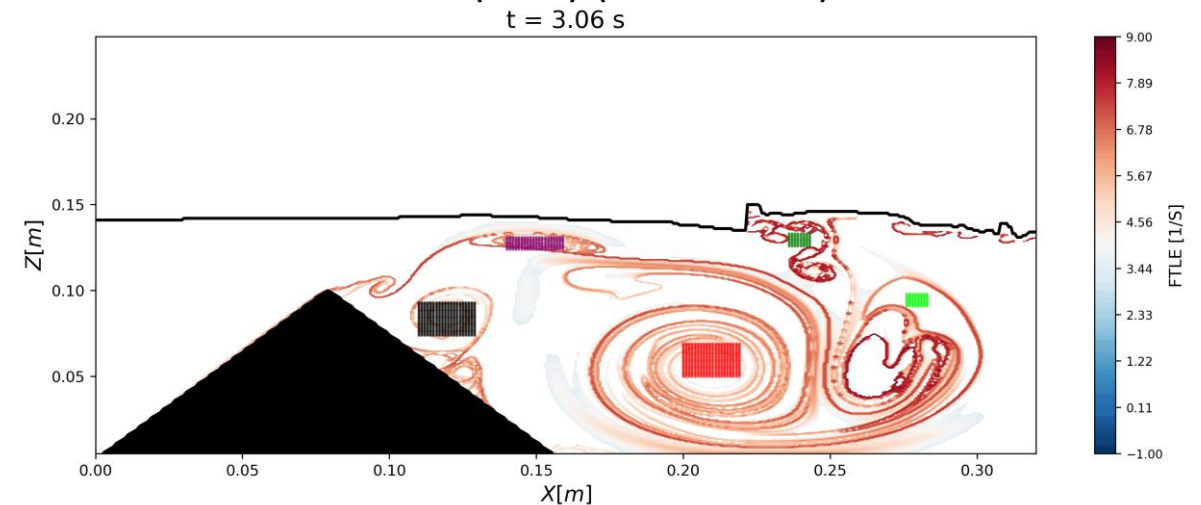
Hernández-Carrasco *et al.*, 2020

High-Frequency Radar (5km) (Upper-Submesoscale)



Rubio *et al.*, 2018

RANS (1cm) (Microscale)

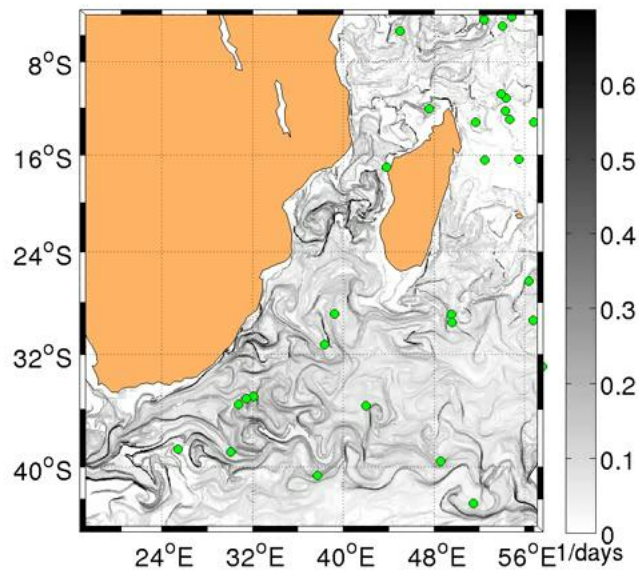


Monsalve *et al.*, 2024

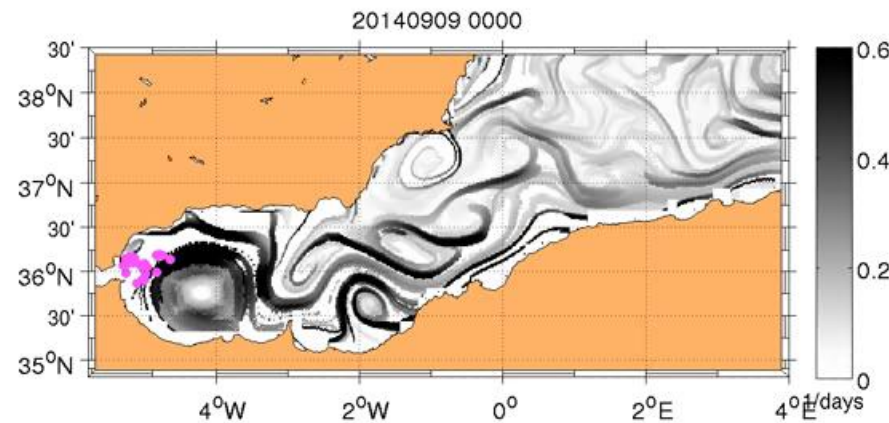
Properties of LCS: Transport barriers

Drifter trajectories + LCS

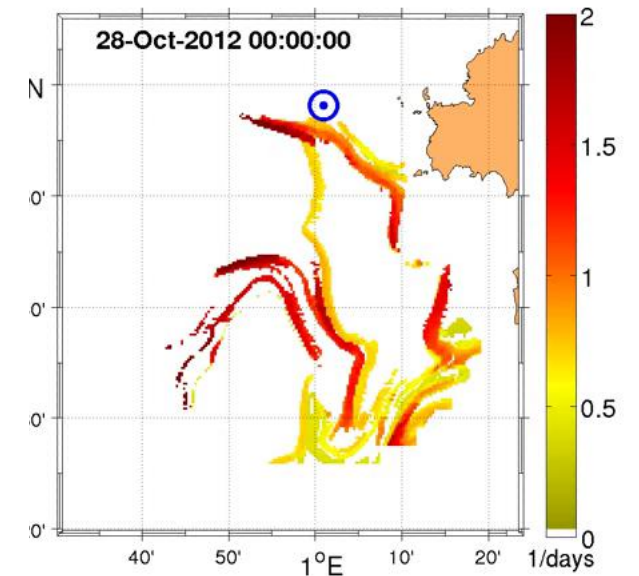
Altimetry (1/8°) (Mesoscale)



Altimetry (1/8°) (Mesoscale)

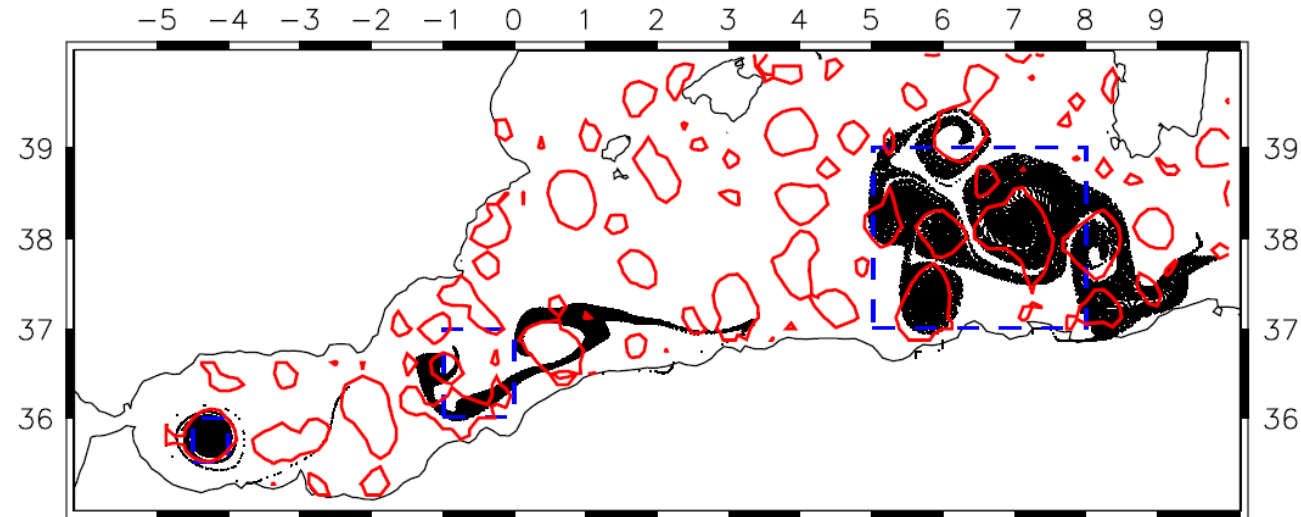


High-Frequency Radar (2km) (Submesoscale)

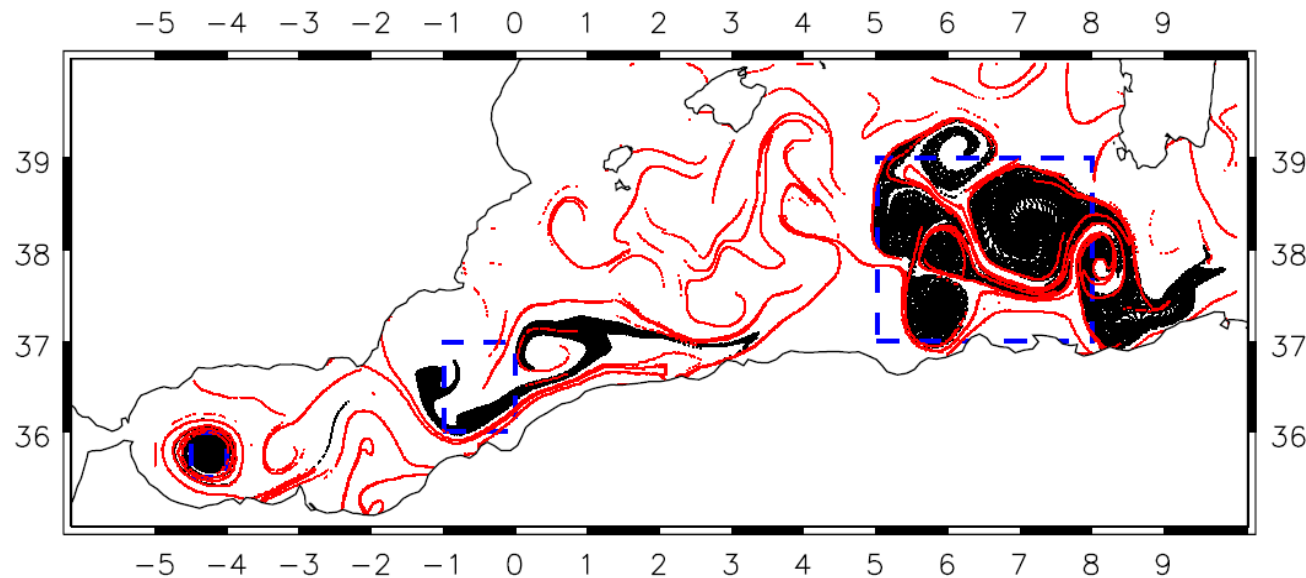


Properties of LCS: Euler Vs. Lagrange

OW (zero contours)



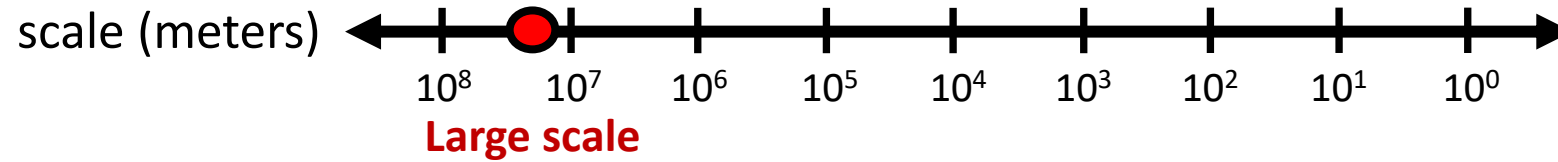
LCS-FSLE



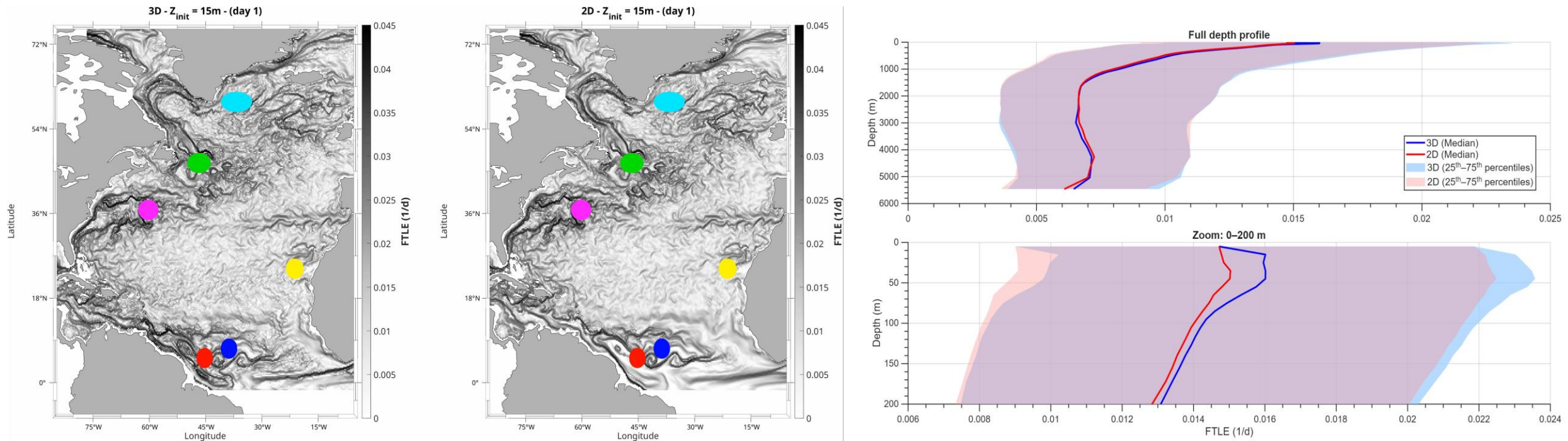
d'Ovidio et al., 2009

Applications of FSLE: Ocean Transport Characterization

Characterization of transport at different scales



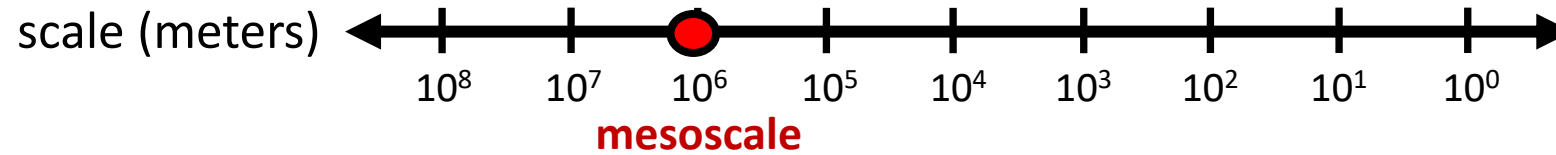
3D-FTLE from ECCO2 (1/4)



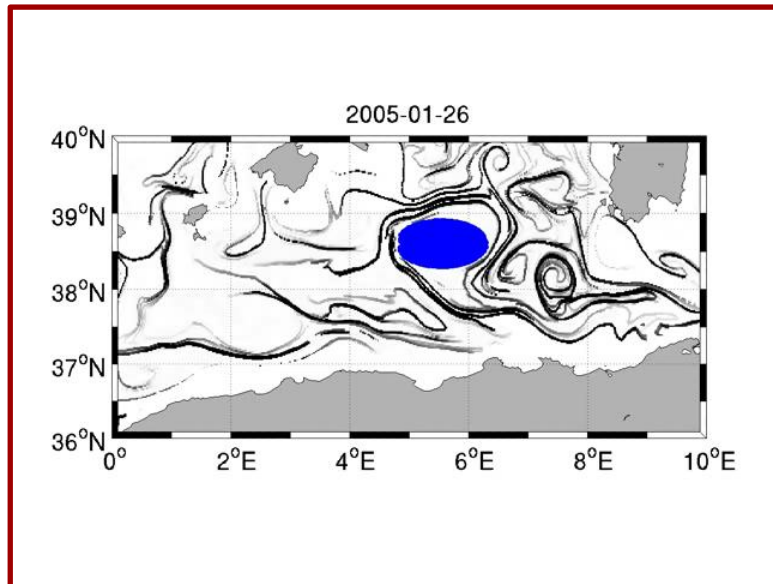
Differences 3D-2D dynamics

Toomey et al., In preparation

Characterization of transport at different scales

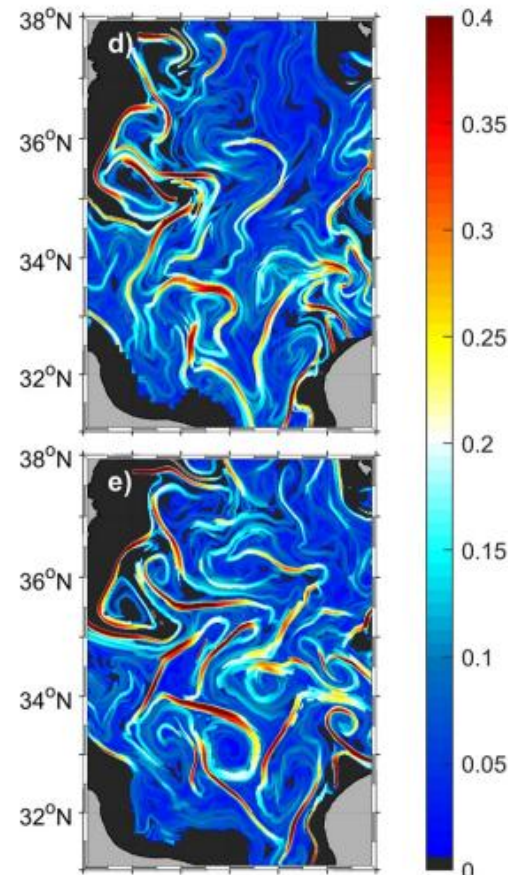


Med- Altimetry 1/8deg + Ekman + Stokes

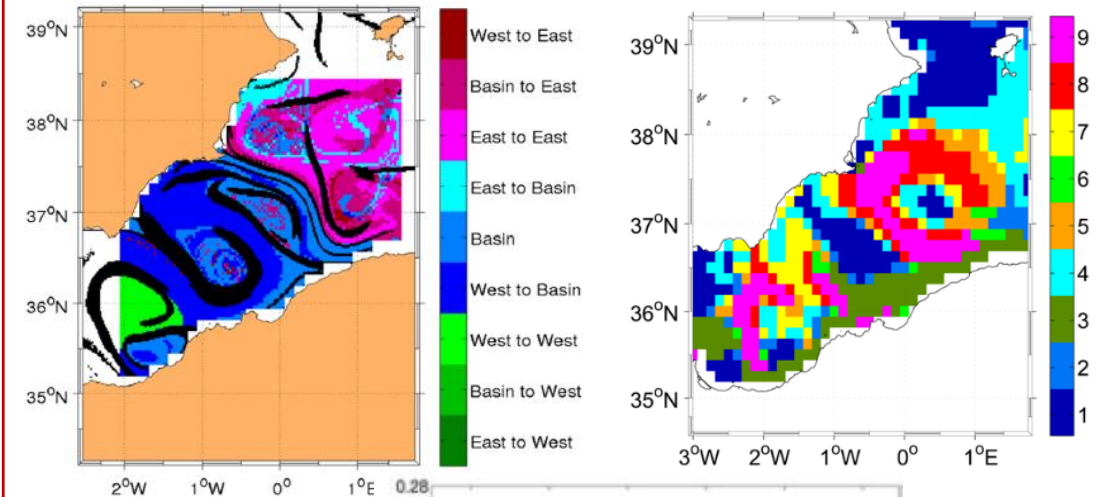


Geostrophic LCS and particle spreading are largely modified by Ekman and Stokes

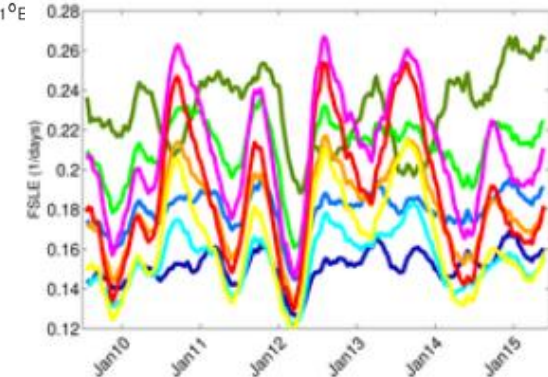
Morales-Marquez et al., 2023



Role of LCS on the regional connectivity

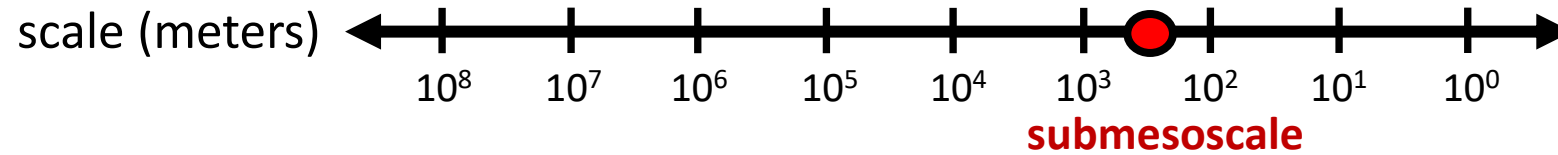


Self-Organizing
Maps analysis



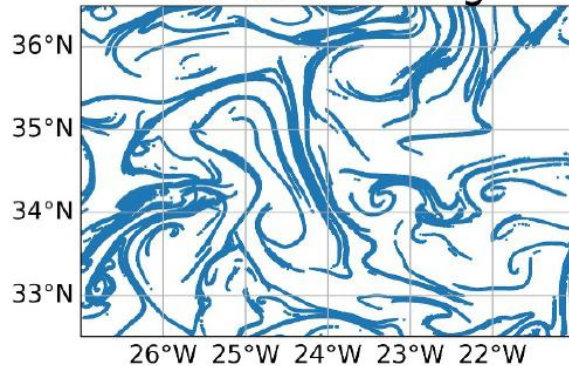
Hernandez-Carrasco and Orfila, 2018, JGR

Characterization of transport at different scales

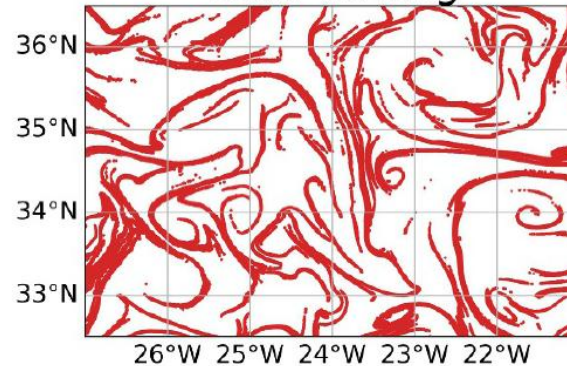


FTLE – ROMS model 1km

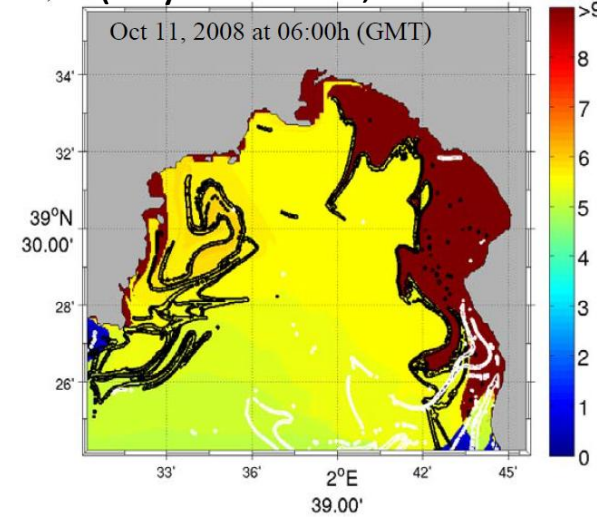
No tidal forcing



Tidal forcing



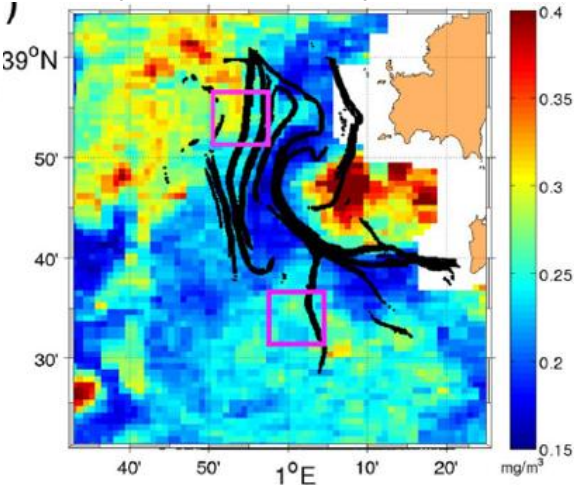
LCS-ROMS 75 m and Residence times (in days)
(Bay of Palma, Balearic Islands)



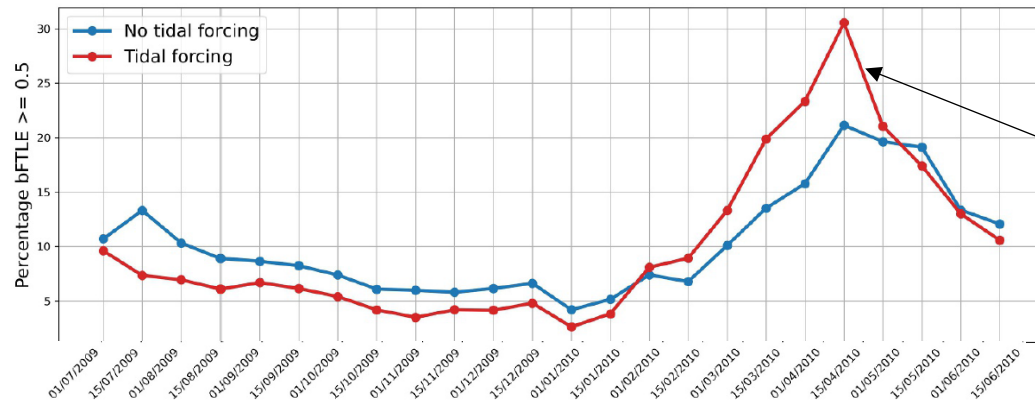
Hernandez-Carrasco et al., 2013

Coastal barriers

HFR-LCS 2km and Chl
(Ibiza Chancel)



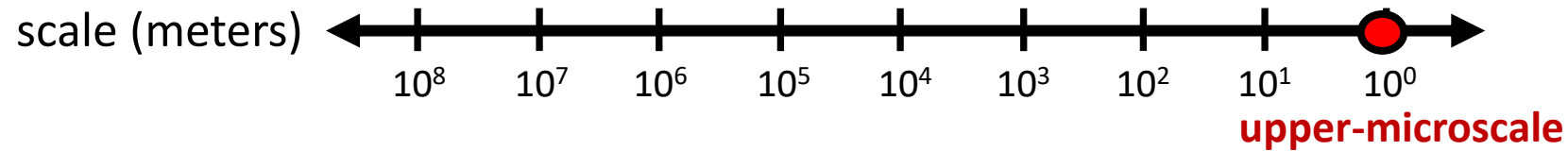
Hernandez-Carrasco et al., 2018



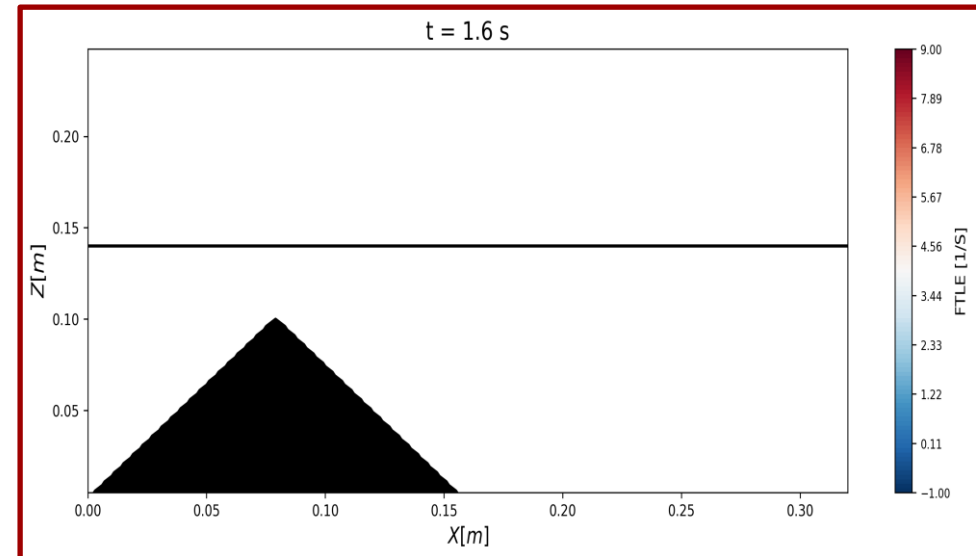
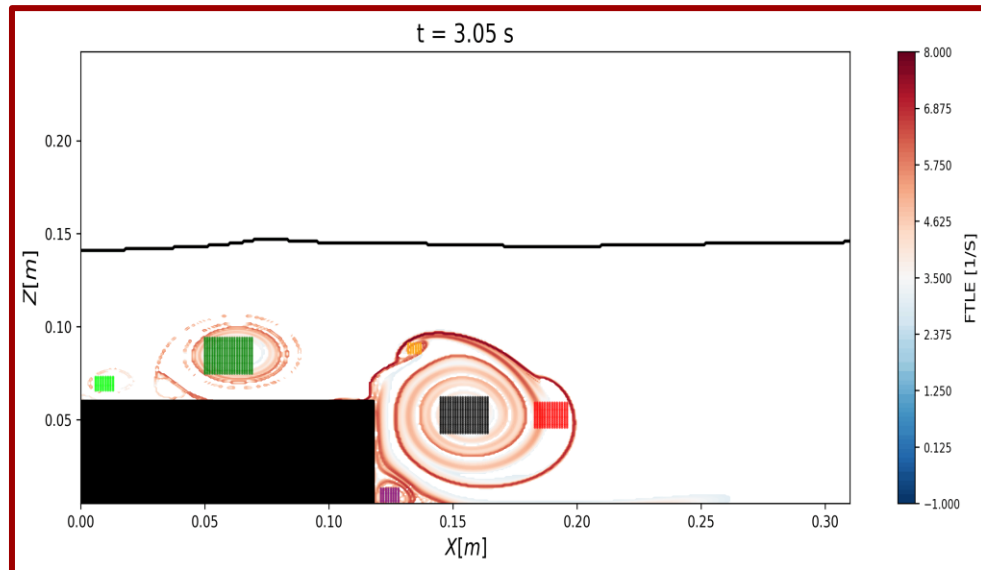
Gomez-Navarro et al., 2026

Tidal induced
Internal waves

Characterization of transport at different scales



Transport patterns induced by the interaction between a solitary wave with submerged objects of different geometry



Wave induced vortex (RANS-VOF)

(Monsalve and Hernandez-Carrasco submitted)

FSLE scaling: Identification of effective dynamical scales

To analyze dispersion in multi-scale flows:
Averaging FSLE over all the particle-pairs at the same spatial scale delta

$$\lambda(\delta, \alpha) = \left\langle \frac{1}{\tau(\delta, \alpha\delta)} \right\rangle \ln \alpha$$

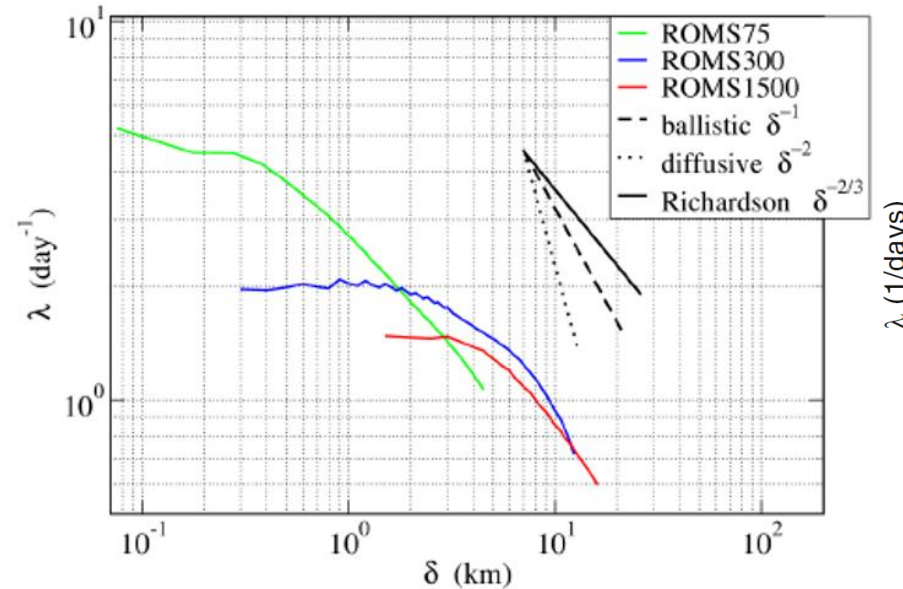
Lyapunov Spectrum, FSLE curves

Dispersion regimes: Richardson, Ballistic, Diffusive

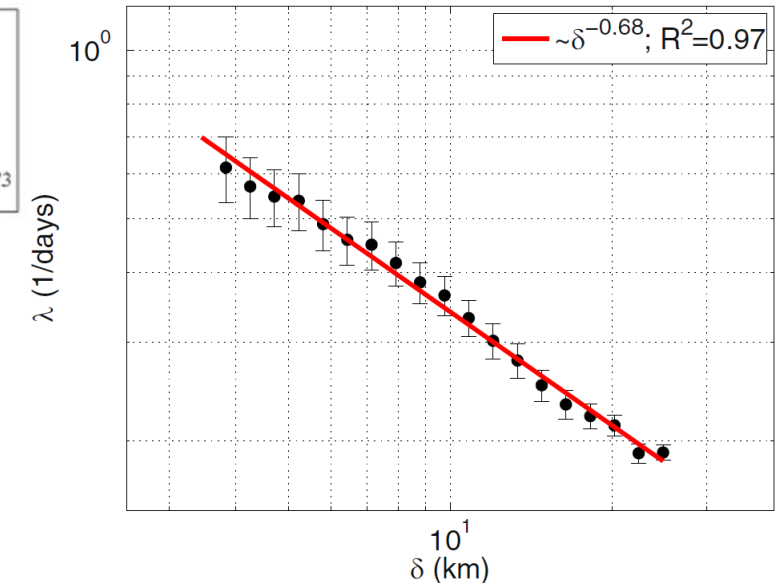
Very useful for:

- Identifying the size of the structures controlling the dispersion
- The assessment of the effective resolution of models/observations
- Quantification of the diffusion scales for optimization/calibration of Lagrangian models

Numerical model (ROMS)



HFR velocity data



Richardson turbulent diffusion:

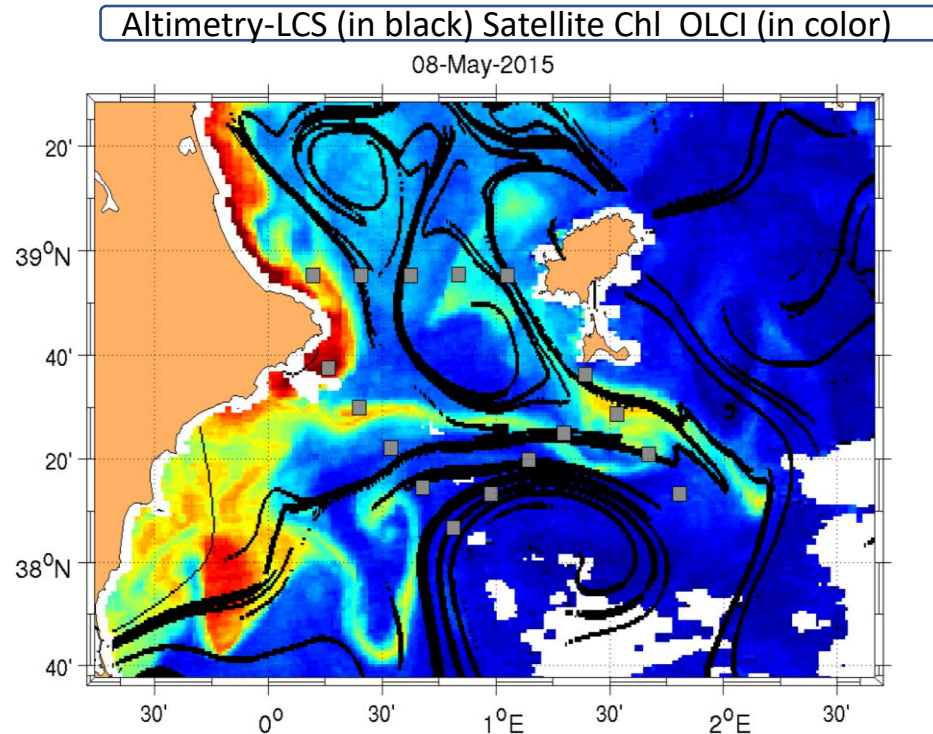
$$\lambda(\delta) \sim \varepsilon^{1/3} \cdot \delta^{-2/3}$$

ε = Mean turbulent dissipation rate

$$2.5 \cdot 10^{-9} \text{ m}^2/\text{s}^3$$

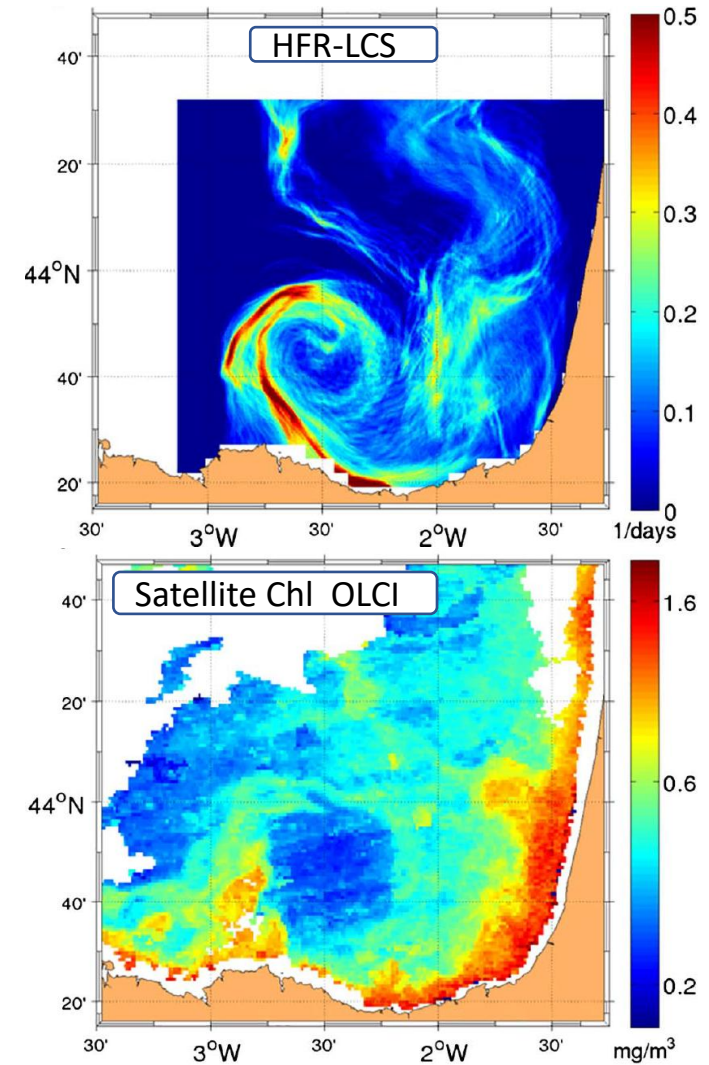
Hernandez-Carrasco et al., SR 2018

Application of FSLE-LCS: Phytoplankton dynamics



Hernandez-Carrasco et al., 2020 PiO

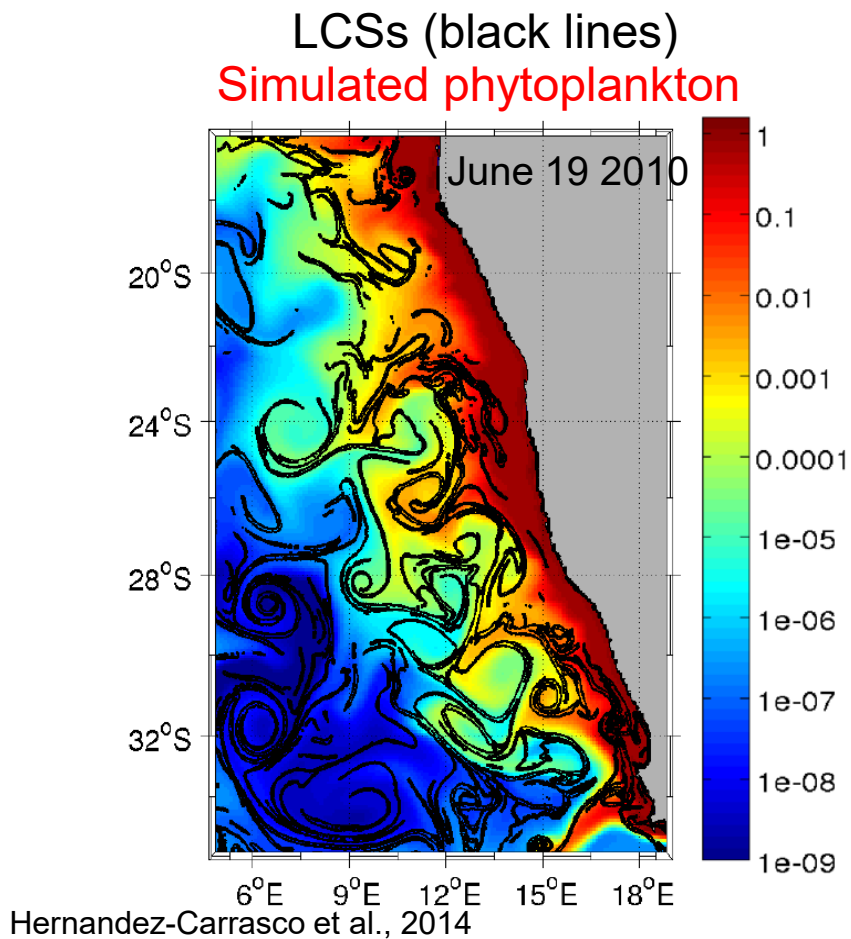
Offshore export of Nutrients/Phytoplankton



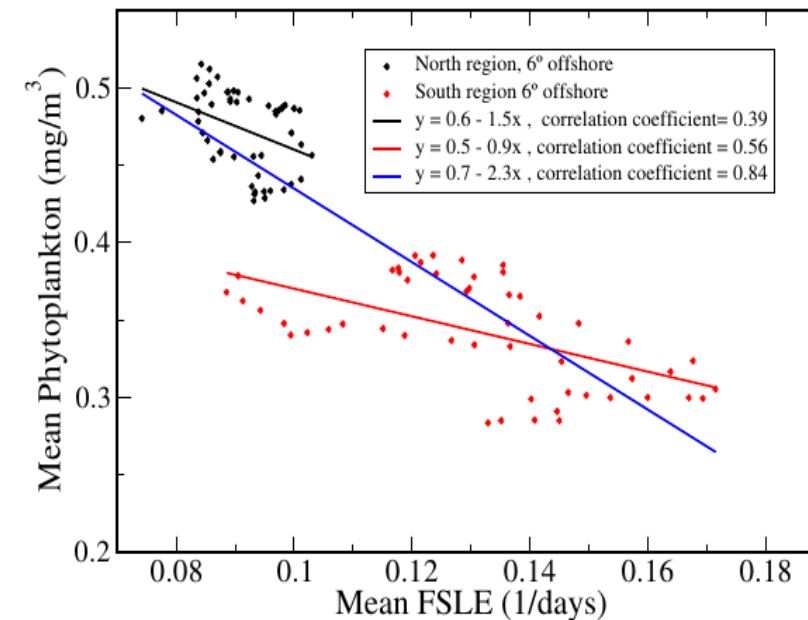
Rubio et al., 2028 RSE

Phytoplankton dynamics

Reduction of phytoplankton biomass induced by high stirring in upwelling regions



Coupled biogeochemical model + ROMS
in Benguela Upwelling System



Inverse relationship between FSLE and Phyto

Outline

1. Lagrangian coherent structures

- Dynamical system approach to Lagrangian transport
- Properties

2. Stretching methods: Finite Size/Time Lyapunov Exponents

- Definition and Properties
- Characterization of ocean transport processes
- Applications: Biological impacts

3. Lagrangian averaged kinematic functions

- Definition and Properties
- Characterization of ocean transport processes.
- Applications: Biological impacts

Lagrangian averaged kinematic functions

$$F_T(\mathbf{r}_0, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} f(\mathbf{R}_t(\mathbf{r}, t_0), t) dt$$

- $F(\mathbf{x}_0, t_0)$ constant over sets invariant under dynamics
- Geometry of the flow
- Relevant dynamical information if f is related to V
- **We monitor fluid parcels accumulating a kinematic property of the flow along its motion.**

Kinematic Eulerian variables: Lagrangian kinematic properties (LKP):
Divergence, Vorticity, EKE

These diagnostics can be used to identify flow structures and their Lagrangian coherence scales

Lagrangian dilation/contraction rate

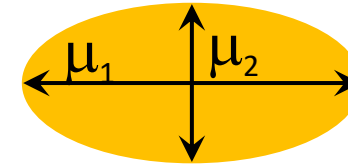
Linearization of
the Flow map $\mathbf{F}$

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{bmatrix}$$

The area of a deformed patch Ω , evolved from Ω_0 , is

$$A = \int \int_{\Omega} d\mathbf{x} = \int \int_{\Omega_0} |\det(\mathbf{F})| d\mathbf{x}_0 = \int \int_{\Omega_0} \mu_1 \mu_2 d\mathbf{x}_0$$

Singular values of $\mathbf{F}$:

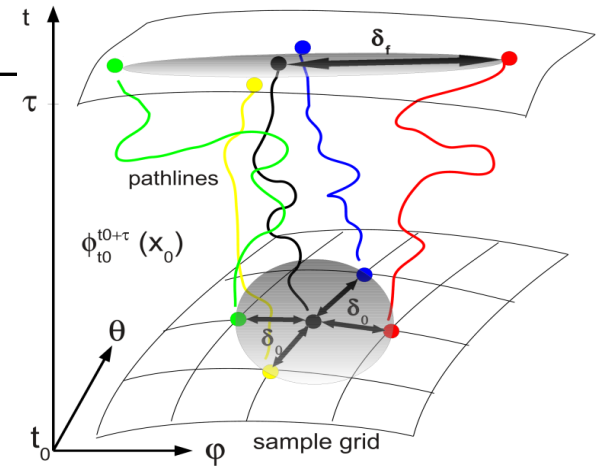


Dilation: $\delta = \mu_1 \mu_2$  Dilation rate: $\Delta = \frac{\log(\delta)}{t}$ (Area changes)

Stretch: $\sigma = \frac{\mu_1}{\mu_2}$  Stretch rate: $\Sigma = \frac{\log(\sigma)}{t}$ (Shape changes)

$$\text{FTLE} = \frac{\log(\mu_1)}{t} \quad \text{FTLE} = \frac{\Delta + \Sigma}{2}$$

FTLE is the combination of area changes and shape changes



Relationship between dilation rate and Eulerian divergence

The material derivative of dilation satisfies: $\frac{D\delta}{Dt} = \delta \nabla \cdot \mathbf{u}$

with the solution: $\delta(t) = \exp \left\{ \int_0^t \nabla \cdot \mathbf{u}[\mathbf{x}(\tau), \tau] d\tau \right\}$

using the definition of dilation rate $\Delta = \frac{\log(\delta)}{t}$

The dilation rate is the average of the horizontal divergence along the trajectories:

Lagrangian divergence

$$\Delta = \frac{1}{t} \int_0^t \nabla \cdot \mathbf{u}[\mathbf{x}(\tau), \tau] d\tau$$

Huntley et al., 2015

Lagrangian Coherent Convergence Structures

$$\delta(\mathbf{r}, t) = \frac{\partial u(\mathbf{r}, t)}{\partial x} + \frac{\partial v(\mathbf{r}, t)}{\partial y}$$

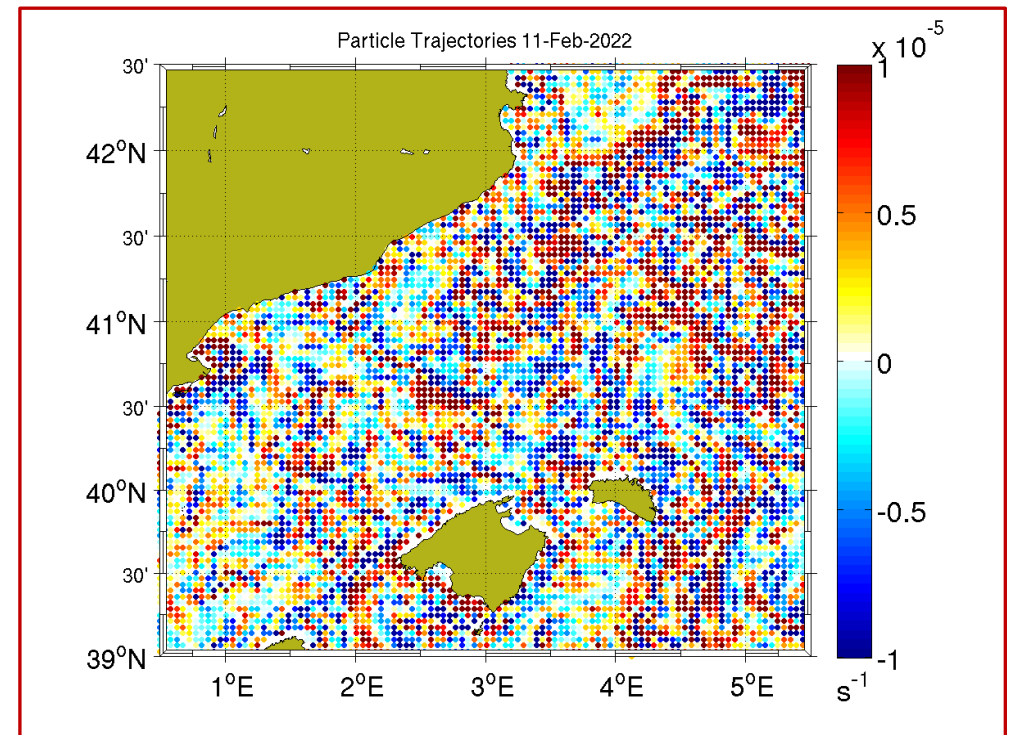
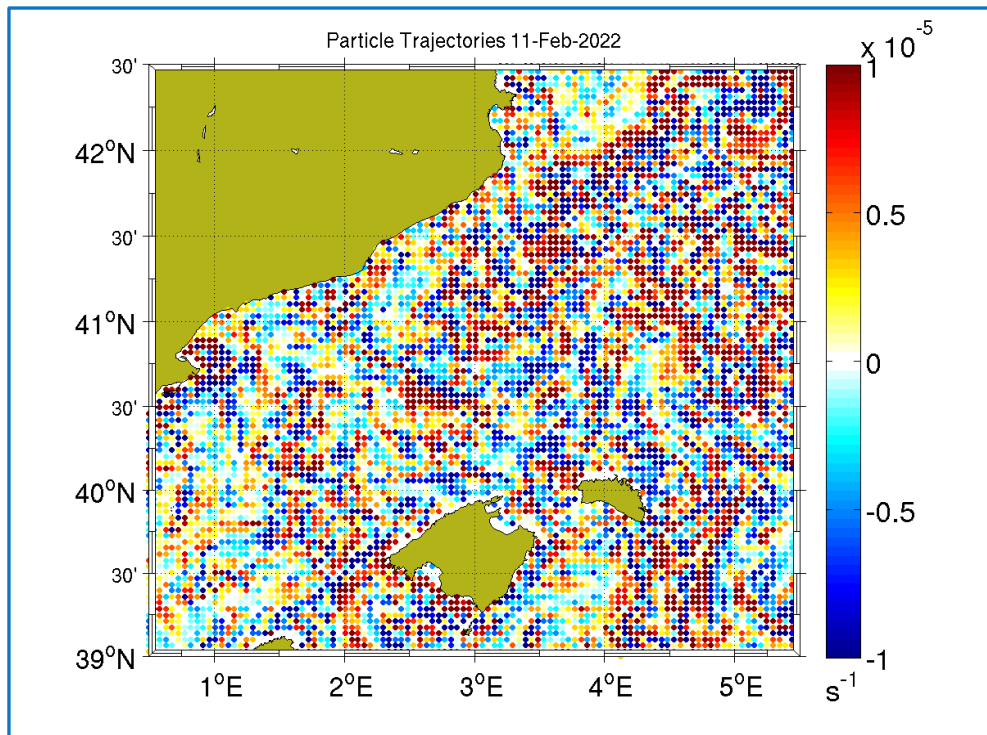
Lagrangian evaluation of divergence along trajectory

$$\delta(\mathbf{R}_t(\mathbf{r}, t_0), t) dt$$

Velocity data:
ROMS, 1.5km, hourly

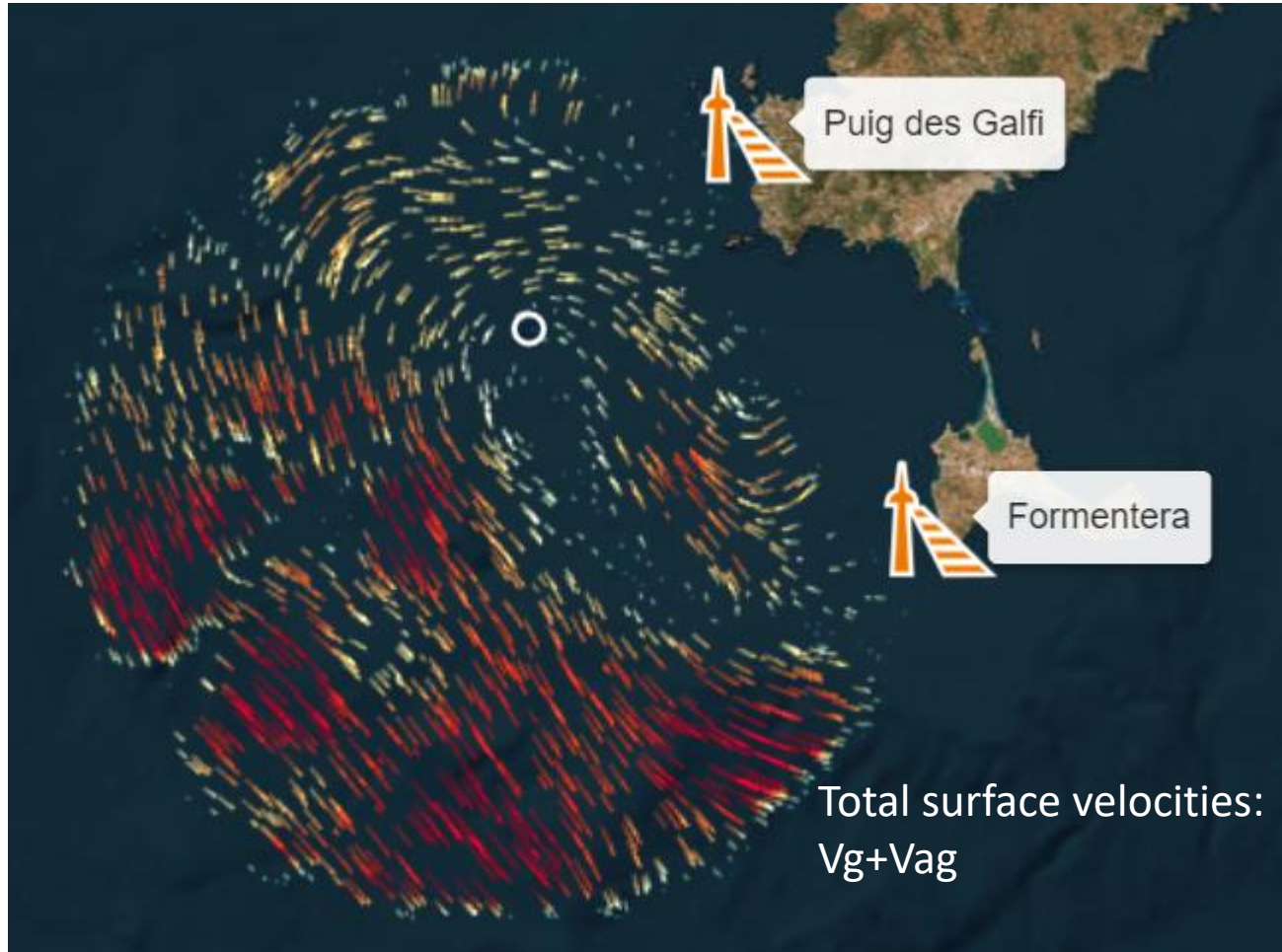
Lagrangian accumulation of the divergence

$$\Delta_T(\mathbf{r}_0, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} \delta(\mathbf{R}_t(\mathbf{r}, t_0), t) dt$$



LD Flow structures: regions of flow dilation/concentration (low/high aggregation)

High-Frequency Radar surface currents

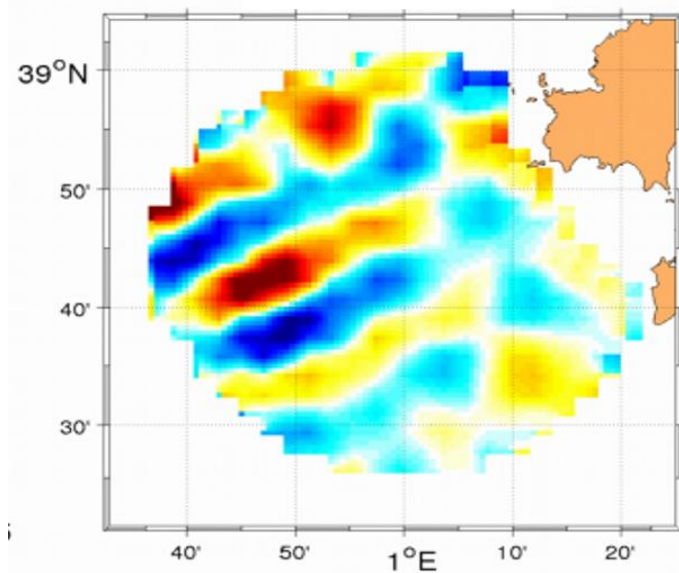


- Hourly velocity fields
- Spatial resolution: 2 km
- Coverage: 90 km
- Operated by SOCIB

Coherent Convergence Flow Structures and Chl concentration

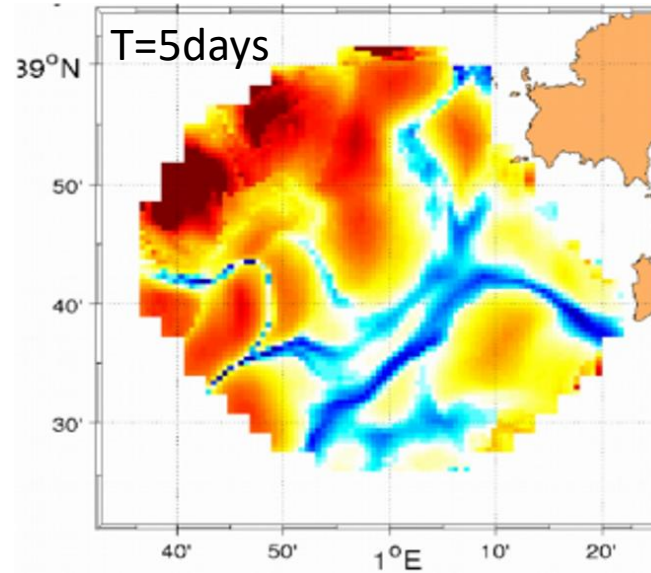
Eulerian divergence

$$\delta(\mathbf{r}, t) = \frac{\partial u(\mathbf{r}, t)}{\partial x} + \frac{\partial v(\mathbf{r}, t)}{\partial y}$$



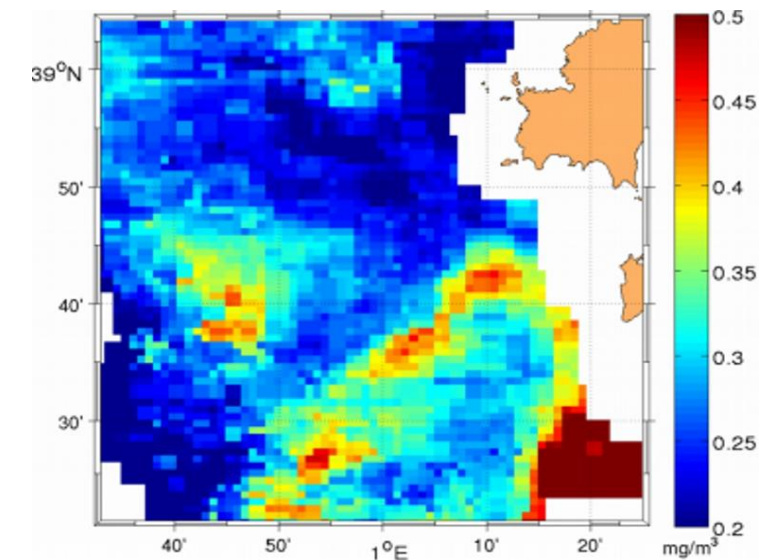
Lagrangian divergence

$$\Delta_T(\mathbf{r}_0, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} \delta(\mathbf{R}_t(\mathbf{r}, t_0), t) dt$$



Phytoplankton concentration

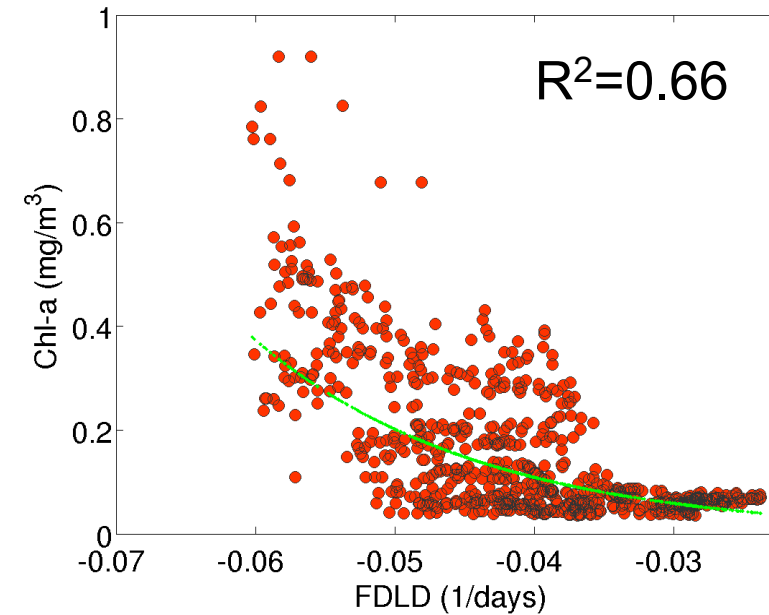
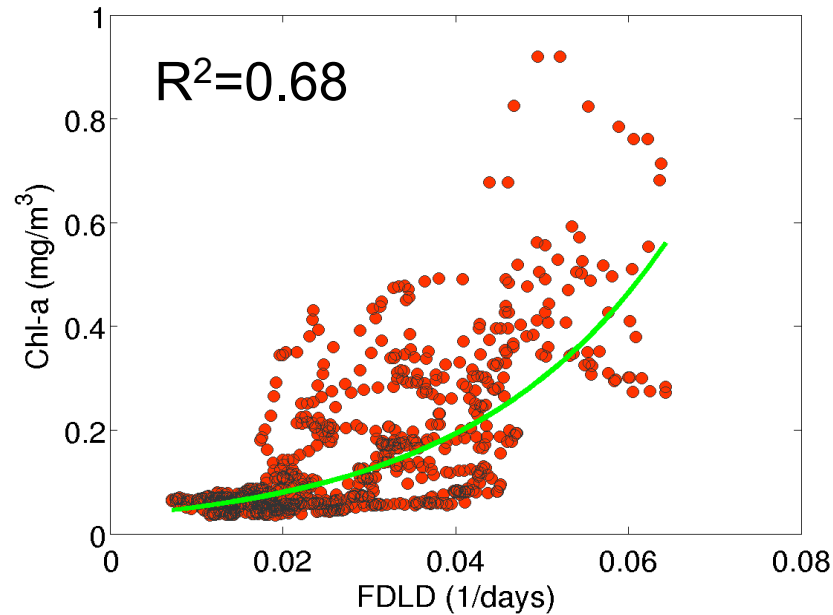
Satellite Chlorophyll
(Globcolour)



Hernandez-Carrasco et al., 2018, *Scientific Reports*

Identify surface persistence divergence/convergence structures
(upwelling/aggregation) with large impact on phytoplankton

Coherent Convergence Flow Structures and Chl concentration



- Good correlations between CHL-FTLD
- Best-fitting: non-linear (polynomial fit of second order)
- There is a coherence threshold above which the flow divergence largely impacts on phytoplankton

Rotationally Coherent Lagrangian Vortex: Supercoherent flow structures

Finite-time coherence filaments

Finite-time Lagrangian relative vorticity

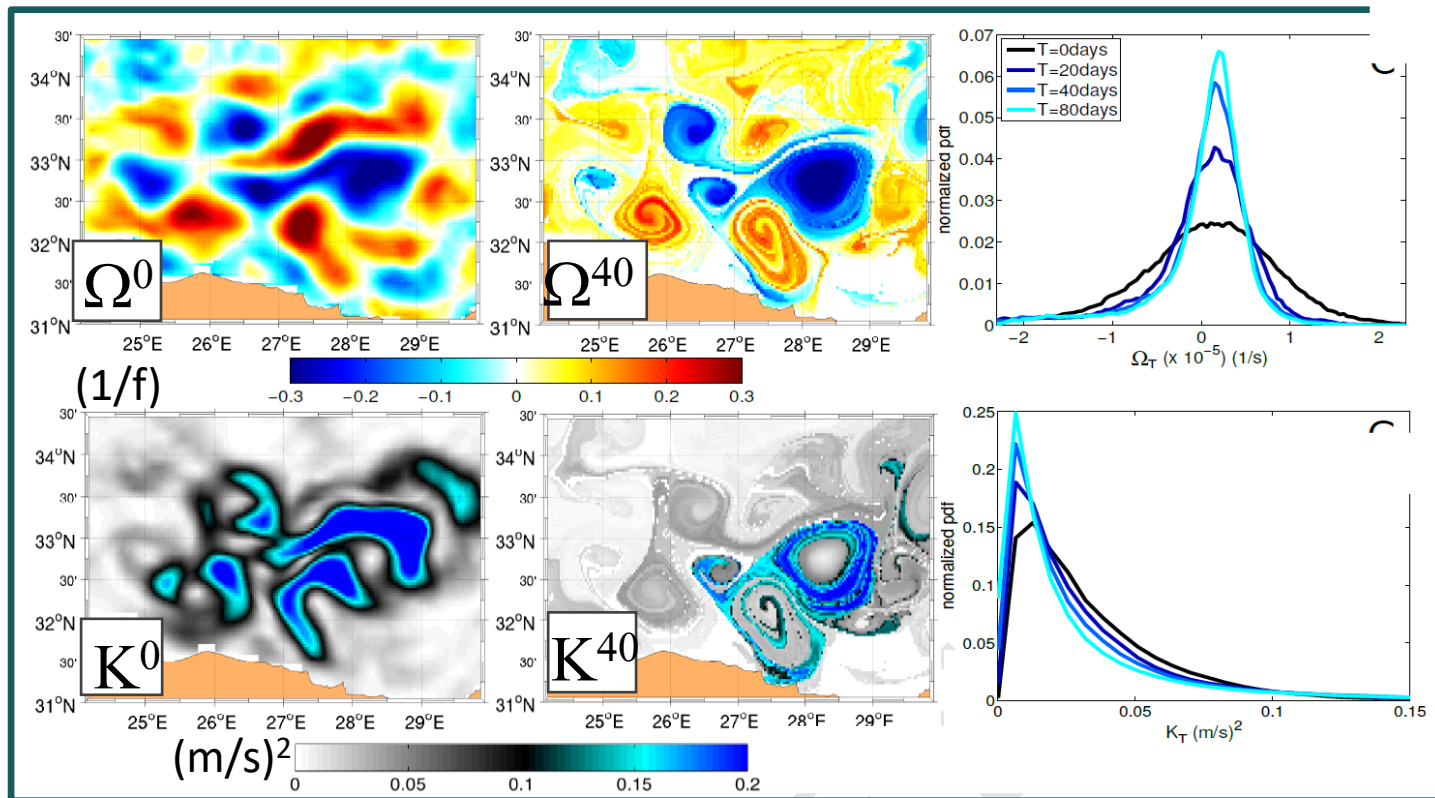
$$\Omega_T(\mathbf{r}_0, t_0, T) = \int_{t_0}^{t_0+T} (\omega(\mathbf{R}_t(\mathbf{r}, t_0), t) - \langle \omega(t) \rangle) dt$$

Velocity data:

Altimetry + Ekman + Stokes
(Morales-Marquez et al, 2021, JGR)

Finite-time Lagrangian eddy kinetic energy

$$K_T(\mathbf{r}_0, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} \kappa(\mathbf{R}_t(\mathbf{r}, t_0), t) dt$$

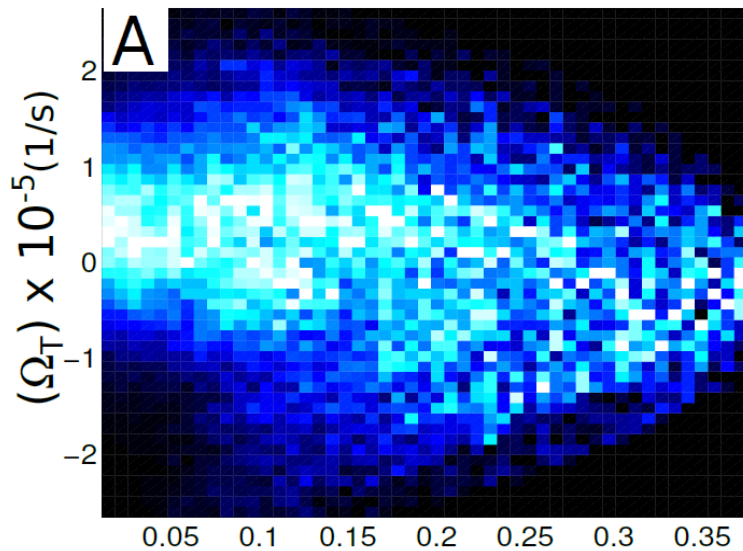


- We monitor fluid parcels accumulating vorticity and turbulent kinetic energy along its motion
- High vorticity and EKE accumulated in filaments
- Time coherence scales of filament formation

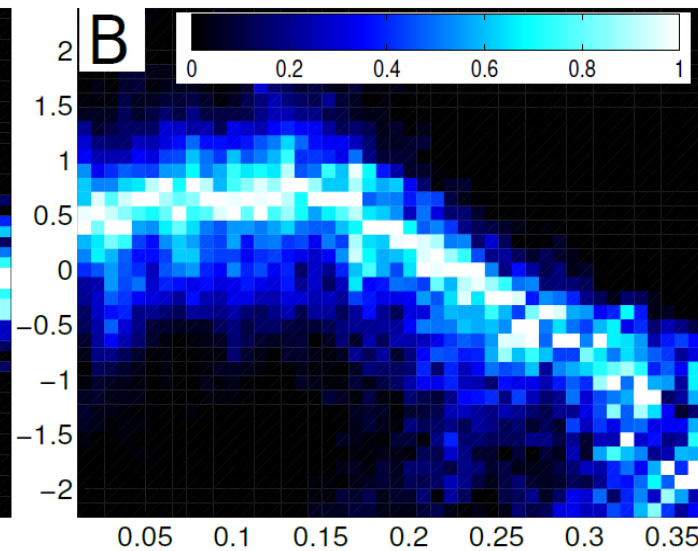
Lagrangian coherence rate

Histograms of Ω_T conditioned to K_T

T=0 days (Eulerian fields)

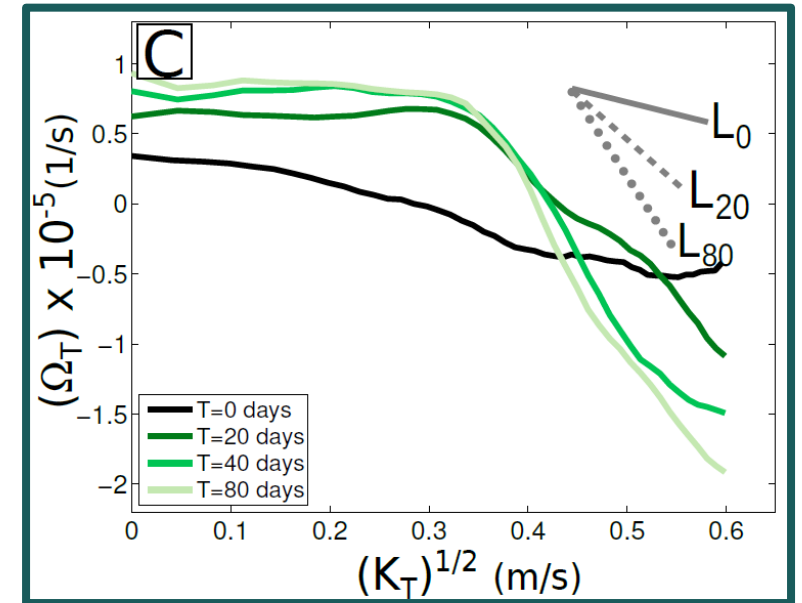


T=40 days



$(K_T) \text{ (m}^2\text{/s}^2\text{)}$

Functional relationships $\Omega_T (K_T)$



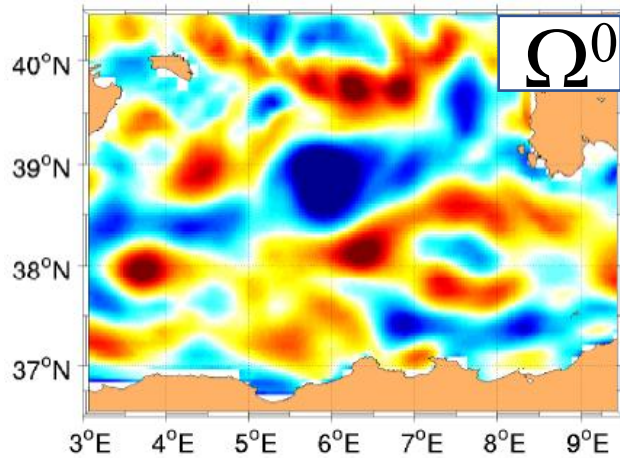
$R^2=0.19$ for T=0 days
 $R^2=0.52$ for T=20 days
 $R^2=0.69$ for T=40 days
 $R^2=0.76$ for T=80 days

$L_0 = 56 \text{ km}$
 $L_{20} = 19 \text{ km}$
 $L_{80} = 10 \text{ km}$

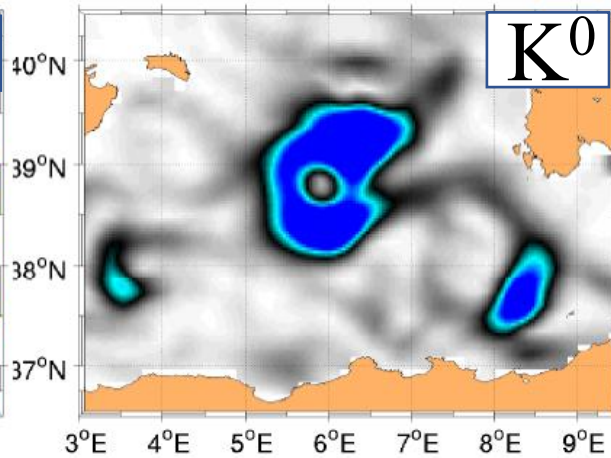
Measure of the range of dynamical coherence time and spatial scales associated with mesoscale ocean features

Identification of highly coherent eddies and diatom patchiness

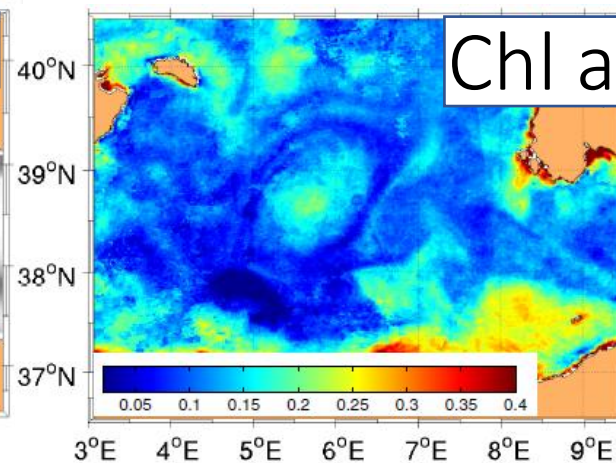
Vorticity



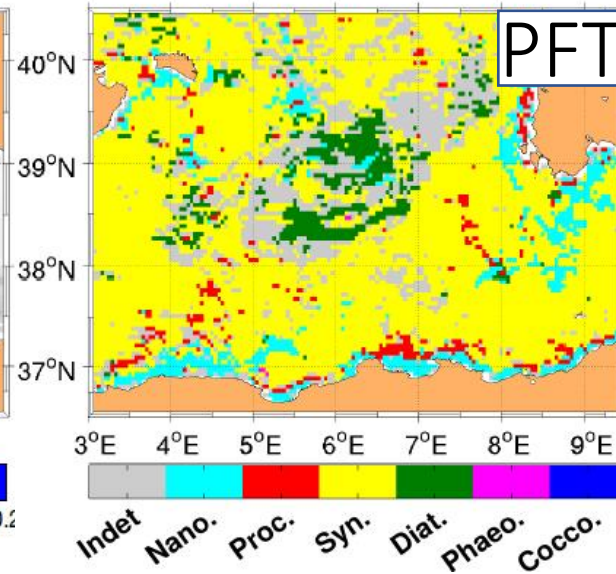
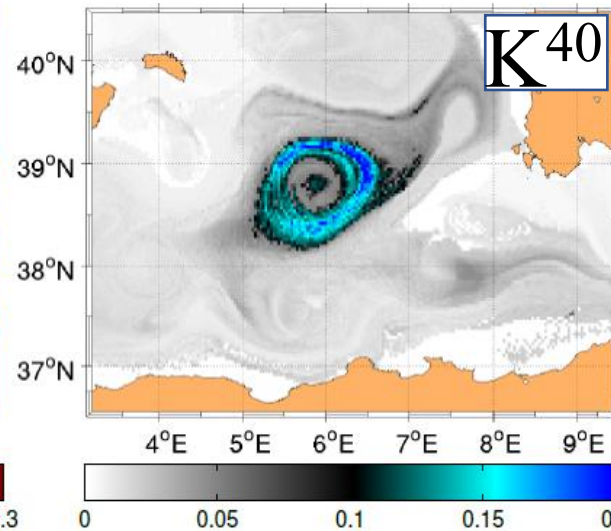
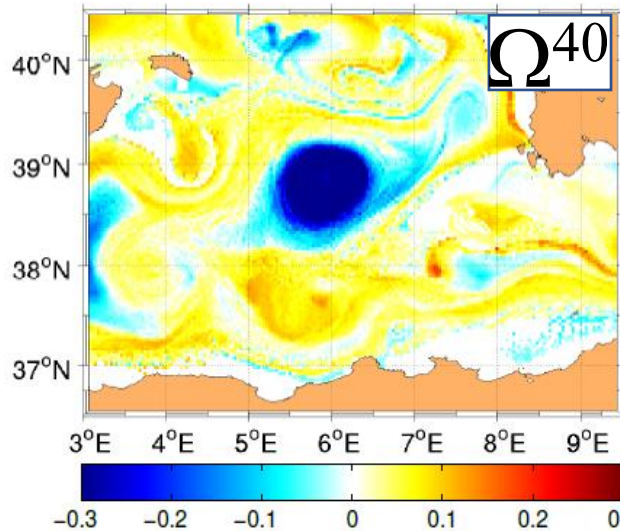
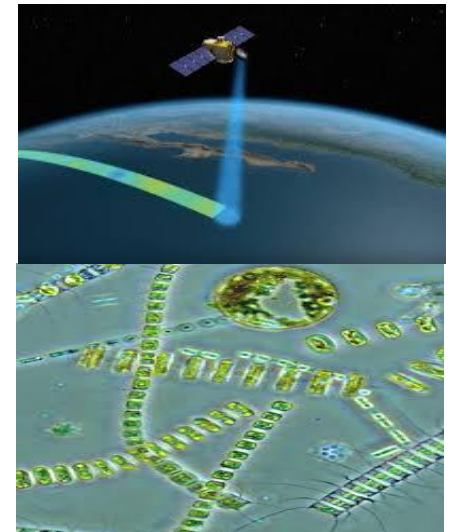
EKE



Phytoplankton

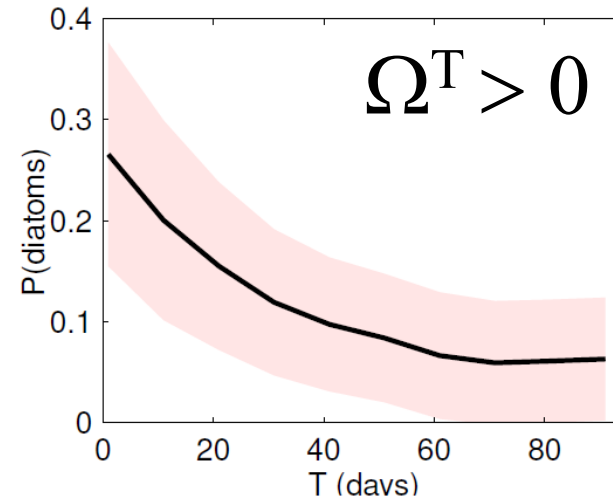
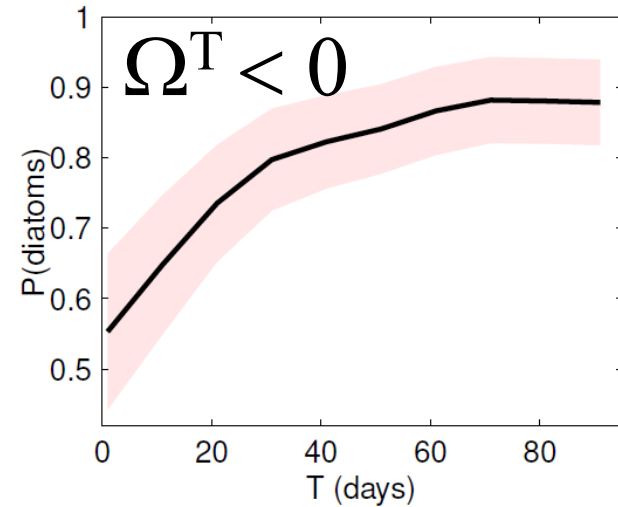


Satellite approach



Phytoplankton Functional Type (PFT) signature in the radiance spectrum, OC-CCI (Navarro et al., 2017, RSE)

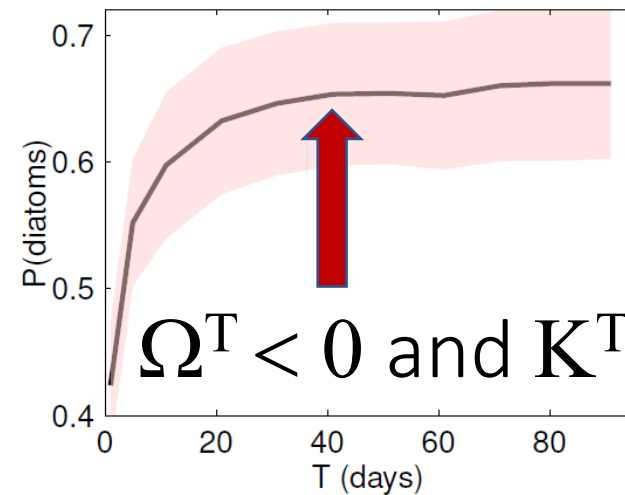
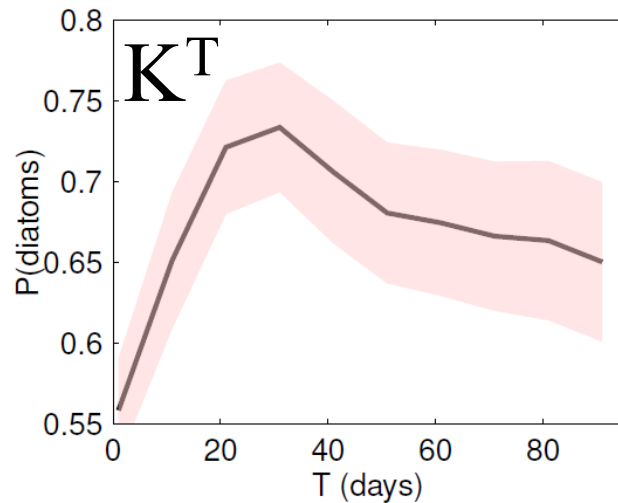
Identification of highly coherent eddies and diatom patchiness



Over 10 years of data

Super-coherent eddies (LRT > 40 days) promote diatoms

Optimal biophysical interaction time scales: > 40 days

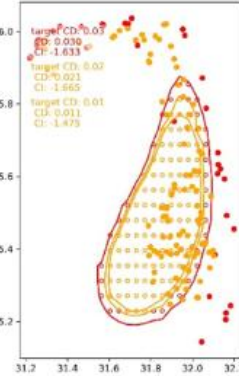
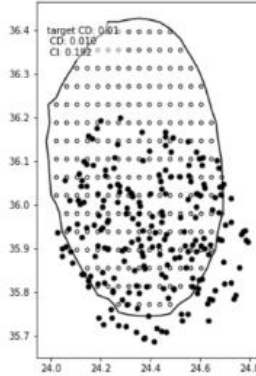
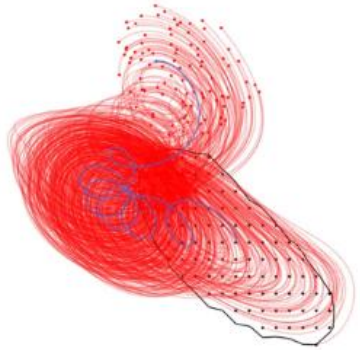


Hernandez-Carrasco et al., 2023, GRL

Super-eddies in the Mediterranean

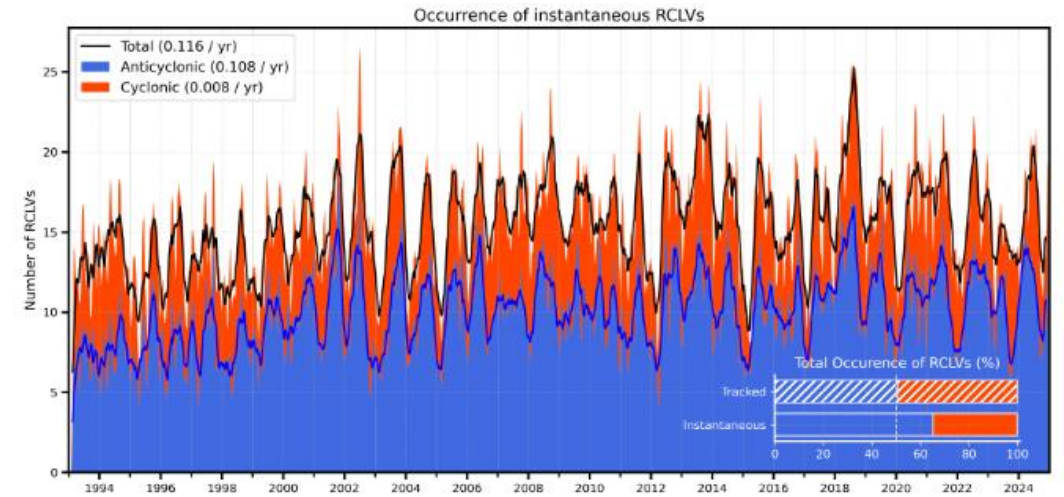
$$\Omega_T(\mathbf{r}_0, t_0, T) = \int_{t_0}^{t_0+T} (\omega(\mathbf{R}_t(\mathbf{r}, t_0), t) - \langle \omega(t) \rangle) dt$$

RCLV
Haller et al, 2016

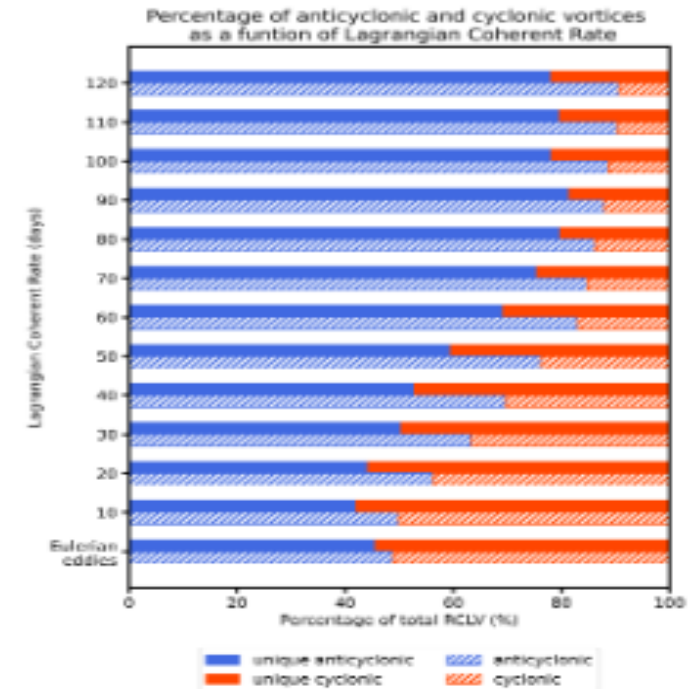


$$CI = \frac{\sigma^2(t_0) - \sigma^2(T)}{\sigma^2(t_0)},$$

$$CD = \frac{A_{\text{convex hull}} - A_{\text{contour}}}{A_{\text{contour}}}$$



Polarity asymmetry



Cortes-Morales and Hernandez-Carraso (in revision)

Limitation of classical residence times

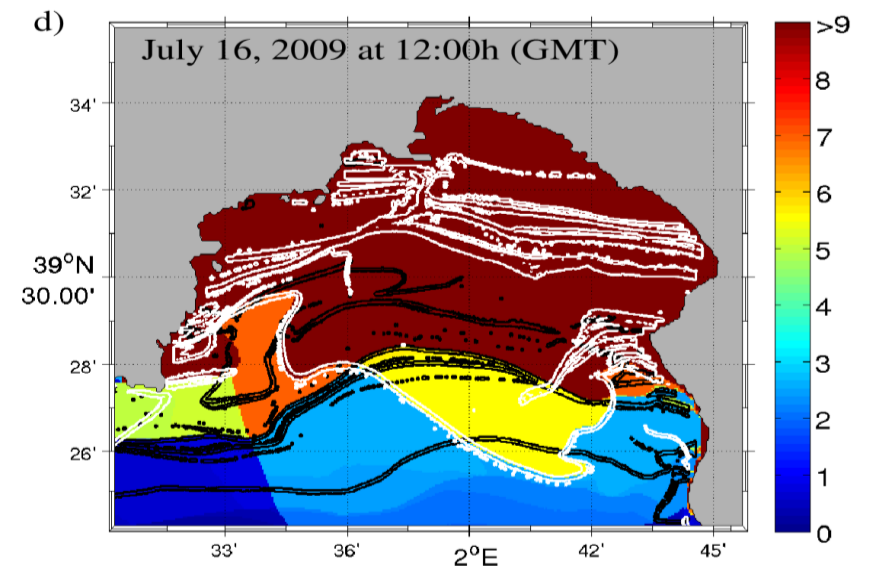
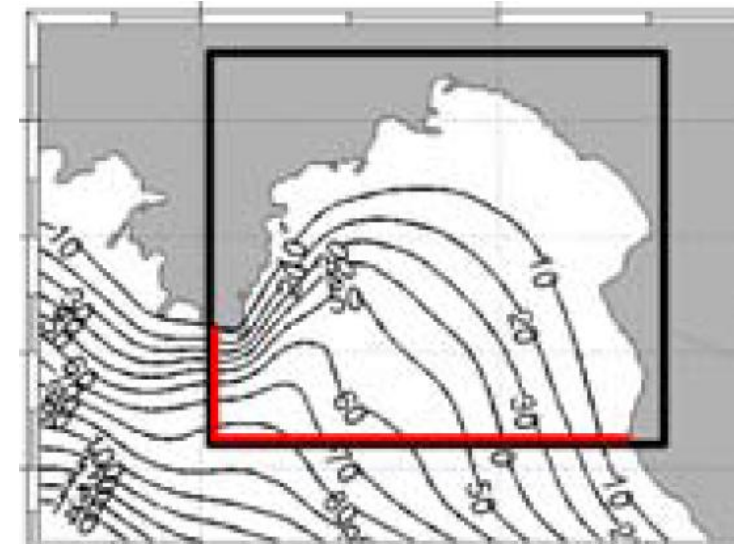
- **Residence, escape and transit times:** Time-scales associated with the rate of fluid particle interchange between a given area and surrounding waters.
- **Synoptic Lagrangian Maps:** The residence time (the time of each fluid parcel remains in a region before crossing a particular boundary) is referenced to its initial position on the grid.

(forward and backward)

$$RT = t_f + t_b$$

- **Limitation:** two-fold dependence on
 - Initial position
 - Common boundaries of the analyzed area

Ambiguous time mixing scales.



Hernandez-Carrasco et al., 2013, *NPG*

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1. Lagrangian coherent structures

- Dynamical system approach to Lagrangian transport
- Properties

2. Stretching methods: Finite Size/Time Lyapunov Exponents

- Definition and Properties
- Characterization of ocean transport processes
- Applications: Biological impacts

3. Lagrangian averaged kinematic functions

- Definition and Properties
- Characterization of ocean transport processes.
- Applications: Biological impacts

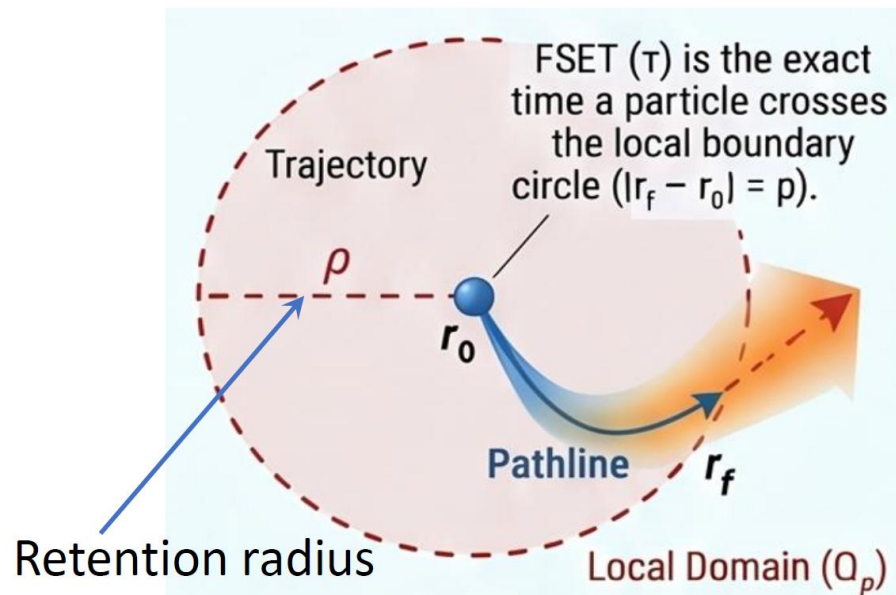
4. Bonus track: Generalized escape rate methods

- Definition and Properties
- Characterization of ocean transport processes.

Local escape time framework

Definition of finite-scale escape time, τ

Defining Local Boundaries



$$\tau = \tau(\mathbf{r}, t, \rho)$$

Time needed to reach a final distance ρ from its initial position in the grid

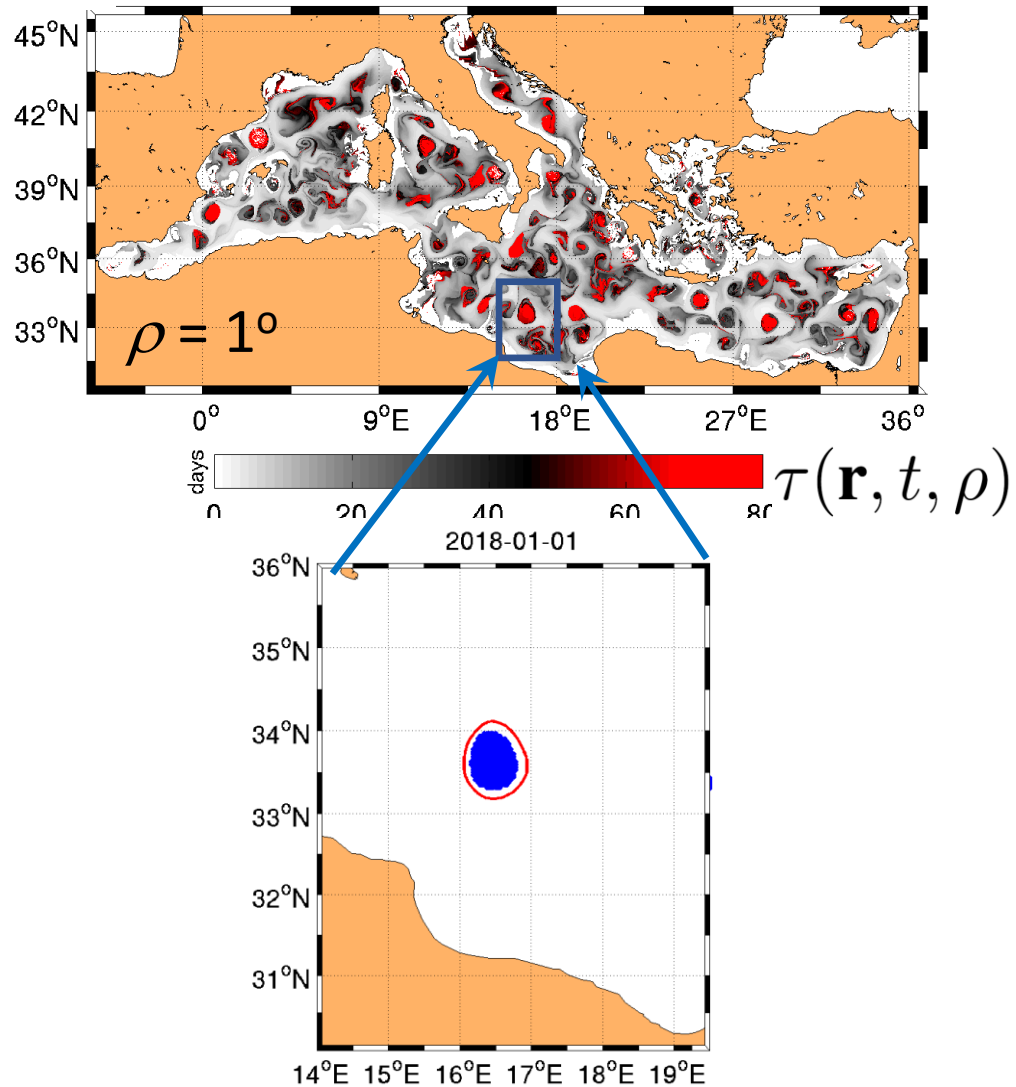
Initial particle position
In 2D: $r = (x, y)$

Initial time of integration

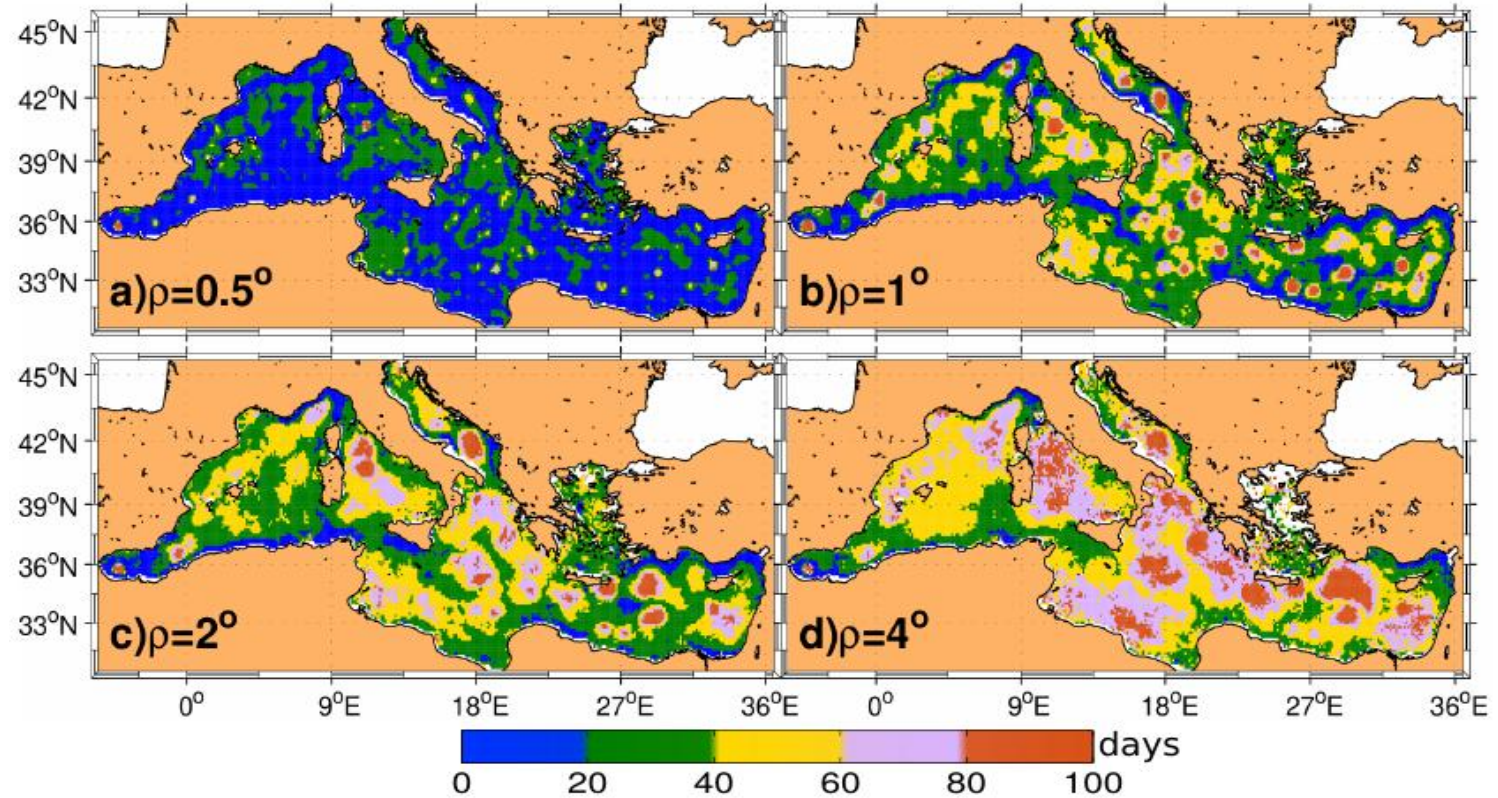
Escape radius

Scale-dependence of FSET

Mesoscale retention structures



Regional/Sub-basin retention



a) $\rho = 0.5^\circ$

c) $\rho = 2^\circ$

b) $\rho = 1^\circ$

d) $\rho = 4^\circ$

Scale-dependence of FSET

Averaged finite-scale escape time:

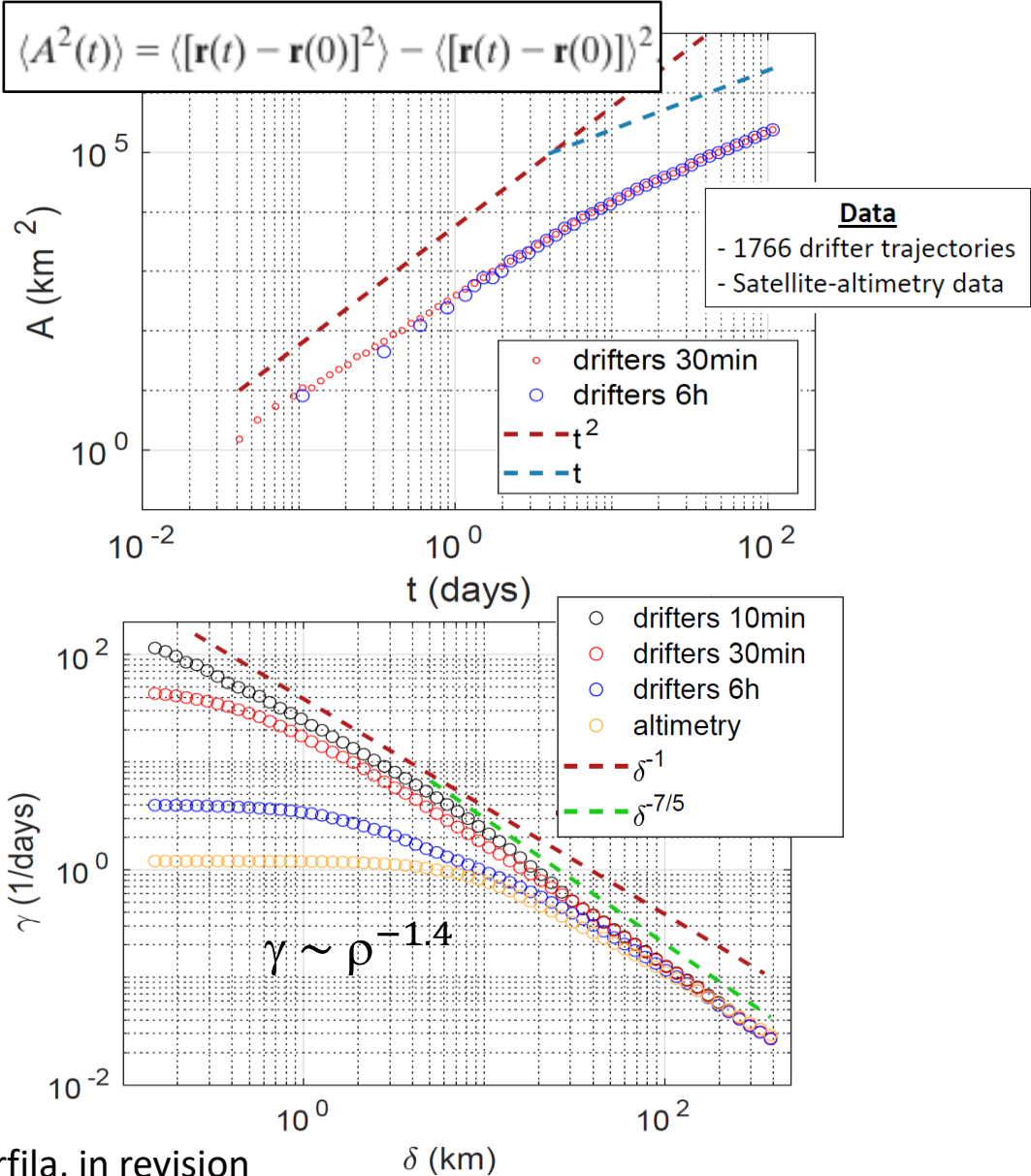
$$\gamma(\mathbf{r}, t, \rho) = \frac{\ln \rho}{|\tau|}$$

$$\gamma(\rho) = \frac{\ln \rho}{\langle |\tau| \rangle}$$

$\langle \tau(\mathbf{r}, t) \rangle_\rho$ = average over all the particles that separate the distance ρ from its initial position.

Single-particle FSLE?
Scale dependent absolute dispersion $A(\rho)$?
Similar to FSLE for the Relative dispersion, but for single-particles.

Ballistic (short times)	$A^2 \sim t^2$	$\gamma \sim \rho^{-1}$
Standard diffusion	$A^2 \sim t$	$\gamma \sim \rho^{-2}$
Intermediate anomalous regime	$A^2 \sim t^{5/3}$ (elliptic area)	$\gamma \sim \rho^{-1.2}$
Intermediate anomalous regime	$A^2 \sim t^{5/4}$ (hyperbolic area)	$\gamma \sim \rho^{-1.6}$



Hernandez-Carrasco and Orfila, in revision

Take-home messages: Why Lagrangian?

- Eulerian perspective only allows to link the characteristics of the biological or physical variable to flow properties at a fixed location (“frozen” in space) and at specific time.
- However, when studying bio-physical interactions in fluid environments any physically-driven change in the biological activity would reflect the cumulative effects of transport dynamics.

Thank you for your attention!!

$$\frac{d\vec{x}}{dt} = \vec{u}(t) = \vec{v}(\vec{x}, t)$$



References on chaotic advection

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